

Multiplicity Distributions in Canonical and Microcanonical Statistical Ensembles

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Abstract

The aim of this paper is to introduce a new technique for calculation of observables, in particular multiplicity distributions, in various statistical ensembles at finite volume. The method is based on Fourier analysis of the grand canonical partition function. Taylor expansion of the generating function is used to separate contributions to the partition function in their power in volume. We employ Laplace's asymptotic expansion to show that any equilibrium distribution of multiplicity, charge, energy, etc. tends to a multivariate normal distribution in the thermodynamic limit. Gram-Charlier expansion allows additionally for calculation of finite volume corrections. Analytical formulas are presented for inclusion of resonance decay and finite acceptance effects directly into the system partition function. This paper consolidates and extends previously published results of current investigation into properties of statistical ensembles.

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I. INTRODUCTION

During the last couple of decades the statistical hadronization model, first introduced by Fermi [1] and Hagedorn [2], has been surprisingly successful in describing fundamental properties of systems created in heavy ion collisions, cosmic rays, and elementary particle reactions. In the context of heavy ion collisions this model has been applied to an extensive set of data on hadron production, ranging from the center of mass energies of the experiments at the SIS, AGS, SPS, and most recently, RHIC facilities. For reviews see [3, 4, 5, 6]. A systematic evolution of thermodynamic parameters as collision energy (and size of colliding ions) is changed [7] has allowed to establish the ‘chemical freeze-out line’, which is now a vital part in our understanding of the phase diagram of strongly interacting matter. More controversially this model has also been applied to a range of elementary collisions [8], where only few particles are produced, and the picture of a gas of hadrons can hardly be suitable. The remarkable ability of the statistical model to explain these data has led to the suggestion [8, 9] that thermal particle production is a general property of the hadronization process itself, rather than the result of a long sequence of microscopic interactions. In this work we will not argue about possible physical interpretations [10] of the partition function of statistical mechanics. However we note that in order to apply a semi-classical approximation a volume of $\mathcal{O}(10 fm^3)$ seems to

be sufficient [11].

One of the answers still outstanding in high energy physics is the one of a possible formation of a deconfined state of matter, where degrees of freedom are quarks and gluons, rather than hadrons, and the nature of the phase transition between these two phases. The growing interest in the study of event-by-event fluctuations in strong interactions is thus motivated by expectations of anomalies in the vicinity of the onset of deconfinement [12] and in the case when the expanding system goes through the transition line between quark-gluon plasma and hadron gas [13]. In particular, a critical point of strongly interacting matter may be accompanied by a characteristic power-law pattern in fluctuations [14]. Multiplicity and charge fluctuations have been indeed proposed to be a good discriminating tool between quark-gluon plasma and hadron gas [15], provided the signal survives the phase transition and subsequent evolution of the system. However, in order to properly assess the discriminating power of such observables, one should firstly calculate fluctuations in a hadron gas by including all known physical effects, such as conservation laws, quantum statistics, resonance decays, kinematical cuts, finite spatial extension, etc.

Only recently, due to a rapid development of experimental techniques, first measurements of fluctuations of particle multiplicities [16] and transverse momenta [17] were performed. And in fact one is tempted to interpret recent NA49 data on multiplicity fluctuations in most central Pb-Pb collisions [18] as the first observation [19] of the recently predicted canonical suppression of fluctuations. The most promising region in the phase diagram for observation of critical phenomena seems to be accessible to the SPS accelerator [20]. A new SPS scan program [21] for different ion sizes as well as center-of-mass energies has been proposed to study strongly interacting systems at different energy densities and life times. This should be as well our main motivation for further investigation of properties of statistical ensembles. The aim is the calculation of ‘base-line’ fluctuations on top of which one hopes to find unambiguous signals of a phase transition [13], a critical point [14], or thermal/chemical (local or global) non-equilibrium [22].

The main subject of the past study has been the mean multiplicity of produced hadrons. However, there is a qualitative difference in the properties of the mean multiplicity and the scaled variance of multiplicity distributions in statistical models. In the case of the mean multiplicity results obtained in the grand canonical ensemble (GCE), canonical ensemble (CE),

and microcanonical ensemble (MCE) approach each other in the large volume limit. One refers here to as the thermodynamical equivalence of statistical ensembles. It was recently found [23, 24] that corresponding results for the scaled variance are different in different ensembles, and thus the choice of ensemble remains a crucial one, even for large systems.

In previous publications [23, 24, 25, 26, 27, 28, 29, 30] the focus was on asymptotic values of the scaled variance, or calculations were altogether only performed in the GCE [31]. In this paper we try to present not only approximations to CE and MCE distributions in the large volume limit, but also to find a reasonable approximation scheme for finite system size in an analytical, rather than Monte Carlo [32], approach. The main subject of this work will be to identify the GCE partition function with the characteristic function of a statistical system. This approach is more effective in terms of mathematical ease as well as in terms of computing time than previous methods. Thus we are presenting for the first time analytical formulas for the final state of a general multi-specie CE or MCE hadron gas at finite volume.

The paper is organized as follows. In Section II the generalized partition function will be introduced and some basic assumptions of this paper will be stated. Section III gives a mathematical approximation to the system partition function in the form of Laplace expansion for the CE, while Section IV is concerned with the MCE formulation of multiplicity distributions in terms of this expansion for a simple quantum gas of massless particles. A general recipe for calculation of the scaled variance is presented. Resonance decay is included into the system partition function in Section V. In Section VI we show a comprehensive comparison of this method to the previously used microscopic correlator approach. A method for finite volume corrections is developed in Section VII. Section VIII will present an application of this method in great detail and aims to give some physical interpretation. A summary in Section IX closes the paper.

II. GENERALIZED GRAND CANONICAL PARTITION FUNCTION

In textbooks on statistical mechanics (see e.g., Ref. [33]) often first the MCE is introduced, where exact conservation laws for energy-momentum and particle number are imposed. Relaxing the constraints for energy and momentum constitutes the CE, while allowing additionally particle number to fluctuate about some mean value introduces the GCE. In a relativistic gas of hadrons quantum numbers (charges) will be the conserved quantities rather than particle

numbers. In this paper it will prove to be of considerable advantage to start off with the GCE formulation and imposing exact conservation laws thereafter. The basic idea is to define the probability of a given number of particles of some species, N_l , at fixed value of conserved charge, Q , i.e. the CE distribution $P(N_l|Q)$, in terms of the GCE distributions, $P(N_l, Q)$ and $P(Q)$. Thus, the GCE partition function will be the basis for all calculations in this work. Generally the (micro)canonical partition function is obtained from the grand canonical one by multiplication with Kronecker (or Dirac) delta-functions which pick out all microstates consistent with a particular conservation law. It is often more economical to use the Fourier representations delta-functions, rather than the delta-function themselves. A short example will motivate the following general treatment. The GCE and the CE partition functions Z and Z^Q are connected as,

$$Z(V, T, \mu_Q) = \sum_{Q=-\infty}^{\infty} \exp\left(\frac{Q\mu_Q}{T}\right) Z^Q(V, T), \quad (1)$$

where V , T , and μ_Q are respectively the system volume, temperature and chemical potential associated with conserved charge Q . The probability of finding the GCE system in a particular charge state Q equals to the number of all states with net-charge Q divided by all accessible states:

$$P(Q) = \frac{\text{all states with charge } Q}{\text{all states}} = \frac{\exp\left(\frac{Q\mu_Q}{T}\right) Z^Q}{Z}. \quad (2)$$

It is important to note that the GCE partition function Z depends on μ_Q/T , while the usual CE partition function does not. We introduce the generalized grand canonical partition function (GGPF) $\mathcal{Z}(\phi_Q)$ by substitution $\mu_Q/T \rightarrow \mu_Q/T + i\phi_Q$ in Z . One then finds:

$$P(Q) = \frac{1}{Z} \int_{-\pi}^{\pi} \frac{d\phi_Q}{2\pi} e^{-iQ\phi_Q} \mathcal{Z}(\phi_Q). \quad (3)$$

The system with one net-charge, Q , could be a quite general one and may include different particle species. In the state of chemical equilibrium the chemical potential of any specie l equals to $\mu_l = q_l \mu_Q$, where q_l is the charge of a particle of specie l . Considering only the distribution of particle species l , the joint probability $P(N_l, Q)$ to find particle number N_l and net-charge Q in the GCE equals to:

$$\begin{aligned} P(N_l, Q) &= \frac{\text{all states with } N_l \text{ particles and charge } Q}{\text{all states}} \\ &= \frac{1}{Z} \int_{-\pi}^{\pi} \frac{d\phi_Q}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi_{N_l}}{2\pi} e^{-iQ\phi_Q} e^{-iN_l\phi_{N_l}} \mathcal{Z}(\phi_Q, \phi_{N_l}). \end{aligned} \quad (4)$$

Similar to Eq. (3) the distribution (4) is presented in terms of the GGPF $\mathcal{Z}(\phi_Q, \phi_{N_l})$ through substitutions $\mu_l/T \rightarrow \mu_l/T + i\phi_{N_l}$, and $\mu_Q/T \rightarrow \mu_Q/T + i\phi_Q$ in the GCE partition function Z . Finally the canonical (or conditional) multiplicity distribution $P(N_l|Q)$ is given by:

$$P(N_l|Q) = \frac{\text{all states with } N_l \text{ particles and charge } Q}{\text{all states with charge } Q} = \frac{P(N_l, Q)}{P(Q)}. \quad (5)$$

Eq. (5) presents the CE particle number distribution $P(N_l|Q)$ in terms of two GCE distributions, $P(N_l, Q)$ and $P(Q)$. A detailed account of this is given in terms of an ideal Boltzmann pion gas in Section VIII.

It is worth noting that Eqs.(2,4,5) are as well the basis for any Monte Carlo approach [32]. A sampling distribution, usually taken from a Boltzmann GCE system, is used to generate a $\{N_l\}$ -tuple of particle multiplicities of all considered species. All ‘events’ consistent with certain constraints, like a set of conserved charges, are accepted, while the rest is rejected. On the basis of this set of all accepted ‘events’ one constructs an ensemble by using a suitable re-weighting scheme to account for quantum statistics and proper normalization. One is now ready to calculate distributions $P(N_l|Q)$, i.e. the conditional distributions to find particle multiplicity N_l in the ‘Monte Carlo ensemble’, while global charge is fixed to Q , and therefore observables like mean multiplicity and multiplicity fluctuations. The advantage of the method presented here is certainly that we proceed in a completely analytical fashion and therefore much unneeded information, like the exact composition of $\{N_l\}$, is ‘integrated away’.

An immediate consequence of Eqs.(2,4,5) is that temperature and chemical potentials appear in our formulation of (micro) canonical distributions (as well as in the Monte Carlo approach [32]). At first sight this seems to be a serious problem and an unnecessary complication of our initial task to find a reasonable approximation to the CE and MCE partition functions. However, the main technical challenge when numerically integrating the original version of the microcanonical partition function arises from a heavily oscillating integrant. Auxiliary parameters T and μ will produce a very smooth function, for which approximation schemes can be used (see Section VIII). In fact chemical potentials and temperature can be factored out, and thus our partition function is just the original partition function times some factor (Appendices B and C). Taking the ratio in Eq.(5) artificially introduced temperature and chemical potential drop out. The quality of our approximation on the other hand will crucially depend on their choice. We will show in Section VII that the requirement of maximizing the GGPF at some given equilibrium point leads to a unique determination of thermal parameters

and moreover constitutes the optimal choice for our approximation scheme. This prescription is self-consistent and can be shown to be not in contradiction to basic thermodynamic relations known from textbooks [33].

Taking the GCE partition function as a basis, we will throughout this paper use the language of statistical mathematics for calculation of conditional multiplicity distributions in any ensemble. So it is worthwhile to give a general outline. For any probability distribution function (PDF) one can define an associated characteristic function (CF) by Fourier back-transformation (Appendix A). In this work we will identify the GGPF as the CF of any GCE distribution, and find joint GCE distributions, e.g. $P(N_l, Q)$, by Fourier analysis of the GGPF. In the example above the CF associated with the PDF $P(N_l, Q)$ in Eq.(4) is $\mathcal{Z}(\phi_Q, \phi_{N_l})/Z$, while the CF associated with the PDF $P(Q)$ in Eq.(2) is $\mathcal{Z}(\phi_Q)/Z$. However, only in the simplest cases one can find an analytical solution for these Fourier integrals. It seems as well to be rather difficult to obtain the CF directly for the PDF $P(N_l|Q)$, and hence we define $P(N_l|Q) = P(N_l, Q)/P(Q)$. Provided one has knowledge of all (sufficiently many) moments of an PDF one can construct the CF and find the exact (approximate) PDF. For practical purposes, however, cumulants, which are obtained by Taylor expansion of the logarithm of the CF, are far better suited.

III. CANONICAL ENSEMBLE

The method presented here will provide the basis for the following discussion. We present in detail our calculation of the CE partition function. In this section we will employ saddle point expansion to approximate the canonical PDF for an ideal hadron resonance gas and focus on the asymptotic solution only. We first calculate the 3-dim distribution of charges in GCE, $P(Q, B, S)$, namely baryon number, strangeness and electric charge. In fact our calculation is similar to that of Becattini *et.al.* [29], however we start explicitly from a GCE partition function which includes chemical potentials. We will identify our version of the CE partition function as the un-normalized probability distribution function of conserved charges in GCE, and show that this type of integral over the GCE partition function generally leads to a Gaussian (multivariate normal) distribution in the large volume limit. In a next step we use this result to obtain the 4-dim PDF for GCE, $P(N, Q, B, S)$, hence the probability to find our GCE hadron-resonance gas in a state (N, Q, B, S) . The ratio is the canonical PDF $P(N|Q, B, S)$, see Eq. (5). In particular we will present a very simple formula for the asymptotic scaled variance of

multiplicity fluctuations.

A. CE Partition Function

The canonical system partition function of a hadron resonance gas with three conserved charges, i.e. electric charge Q , baryon number B , and strangeness S is given by the triple Fourier integral over GCE partition function [34]:

$$\mathcal{Z}^{Q,B,S} = \int_{-\pi}^{\pi} \frac{d\phi_Q}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi_B}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi_S}{2\pi} e^{-iQ\phi_Q} e^{-iB\phi_B} e^{-iS\phi_S} \times \exp \left[\sum_l z_l(\phi_Q, \phi_B, \phi_S) \right]. \quad (6)$$

Adopting vector notation for the set of conserved charges $(Q, B, S) = \vec{Q} = Q^j$ and angles $(\phi_Q, \phi_B, \phi_S) = \vec{\phi} = \phi_j$ we can write Eq.(6) in compact form:

$$\mathcal{Z}^{Q^j} = \left[\prod_{j=1}^3 \int_{-\pi}^{\pi} \frac{d\phi_j}{(2\pi)} \right] e^{-iQ^j \phi_j} \exp \left[\sum_l z_l(\phi_j) \right], \quad (7)$$

where repeated upper and lower indexes j imply summation over j . The single particle partition function of particle specie l is given by:

$$z_l(\phi_j) = \frac{g_l V}{(2\pi)^3} \int d^3p \ln \left(1 \pm e^{-(\varepsilon_l - \mu)/T} e^{i q_l^j \phi_j} \right)^{\pm 1} \equiv V \psi_l(\phi_j), \quad (8)$$

where we have introduced particle l 's quantum number configuration $q_l^j = \vec{q}_l = (q_l, b_l, s_l)$, it's degeneracy factor $g_l = (2J_l + 1)$, internal angular momentum J_l , mass m_l , and energy $\varepsilon_l = \sqrt{p^2 + m_l^2}$, the chemical potential vector $\mu^j = (\mu_Q, \mu_B, \mu_S)$, and particle l 's chemical potential $\mu_l = q_l^j \mu_j$. V is the system volume, and T it's temperature. The summation \sum_l includes also anti-particles, for which $q_l^j \rightarrow -q_l^j$. The (Wick rotated) fugacity of particle specie l is thus given by the substitution $\lambda_l = \exp[q_l^j \mu_j / T] \rightarrow \lambda_l = \exp[q_l^j (\mu_j / T + i\phi_j)]$. The upper sign in the Eq. (8) denotes Fermi-Dirac statistics (FD), while the lower is used for Bose-Einstein statistics (BE). The case of Maxwell-Boltzmann statistics (MB) is analogous and will not be discussed separately. Together with $\sum_l \psi_l(\phi_j) = \Psi(\phi_j)$ the system partition function can be written as:

$$\mathcal{Z}^{Q^j} = \left[\prod_{j=1}^3 \int_{-\pi}^{\pi} \frac{d\phi_j}{(2\pi)} \right] e^{-iQ^j \phi_j} \exp \left[V \Psi(\phi_j) \right]. \quad (9)$$

The following calculation is based only on the general form of Eq.(9), and can easily be generalized to a larger set of conserved quantities. In the language of statistical mathematics the function $\Psi(\phi_j)$ would be called cumulant generating function (CGF) of the partition function \mathcal{Z}^{Q_j} , while $\exp[V \Psi(\phi_j)]$ is called CF. For large volume, $V \rightarrow \infty$, the main contribution to the integral Eq.(9) comes from a small region around the origin [29]. Thus we proceed by Taylor expansion of the CGF $\Psi(\phi_j)$ around $\phi_j = \vec{0}$ and introduce cumulant tensors κ :

$$\kappa_n^{j_1, j_2, \dots, j_n} \equiv (-i)^n \left. \frac{\partial^n \Psi(\phi_j)}{\partial \phi_{j_1} \partial \phi_{j_2} \dots \partial \phi_{j_n}} \right|_{\phi_j = \vec{0}}. \quad (10)$$

The CGF therefore can be expressed in terms of a Taylor series:

$$\Psi(\phi_j) \simeq \sum_{n=0}^{\infty} \frac{i^n}{n!} \kappa_n^{j_1, j_2, \dots, j_n} \phi_{j_1} \phi_{j_2} \dots \phi_{j_n}, \quad (11)$$

where summation over repeated indices is implied. The cumulant of 0^{th} order is just the logarithm of the GCE partition function Z divided by the volume, $Z \equiv \exp(V \kappa_0)$. Hence, after extending the limits of integration to $\pm\infty$, which will introduce a negligible error, we find:

$$\mathcal{Z}^{Q_j} \simeq Z \left[\prod_{j=1}^3 \int_{-\infty}^{\infty} \frac{d\phi_j}{(2\pi)} \right] \exp \left[-iQ^j \phi_j + V \sum_{n=1}^{\infty} \frac{i^n}{n!} \kappa_n^{j_1, j_2, \dots, j_n} \phi_{j_1} \phi_{j_2} \dots \phi_{j_n} \right]. \quad (12)$$

It is worth noting that the integrant of Eq.(12) is not 2π -periodic anymore, while the one of Eq.(9) is. Spelling out the first two terms of the summation yields:

$$\begin{aligned} \mathcal{Z}^{Q_j} \simeq Z \left[\prod_{j=1}^3 \int_{-\infty}^{\infty} \frac{d\phi_j}{(2\pi)} \right] \exp \left[-iQ^j \phi_j + -iV \kappa_1^j \phi_j - V \frac{\kappa_2^{j_1, j_2}}{2!} \phi_{j_1} \phi_{j_2} \right. \\ \left. + V \sum_{n=3}^{\infty} \frac{i^n}{n!} \kappa_n^{j_1, j_2, \dots, j_n} \phi_{j_1} \phi_{j_2} \dots \phi_{j_n} \right]. \end{aligned} \quad (13)$$

Performing now a change of variables will simplify this triple integral.

$$\theta_j = \sqrt{V} \sigma_j^k \phi_k, \quad (14)$$

where σ_j^k is the square root of the second rank tensor κ_2 :

$$\sigma_j^k \equiv \left(\kappa_2^{1/2} \right)_j^k. \quad (15)$$

The element $d\theta_j$ equals to:

$$d\theta_j = \det |\sqrt{V} \sigma| d\phi_j = V^{3/2} \det |\sigma| d\phi_j. \quad (16)$$

Lastly in terms of this transformation normalized cumulant tensors λ are introduced:

$$\lambda_n^{j_1, j_2, \dots, j_n} \equiv \kappa_n^{k_1, k_2, \dots, k_n} (\sigma^{-1})_{k_1}^{j_1} (\sigma^{-1})_{k_2}^{j_2} \dots (\sigma^{-1})_{k_n}^{j_n} . \quad (17)$$

The new variable ξ^j will be a measure for the distance of the actual charge vector Q^j to the peak of the distribution of the PDF:

$$\xi^j = (Q^k - V \kappa_1^k) (\sigma^{-1})_k^j V^{-1/2} . \quad (18)$$

Including all these steps at once yields:

$$\mathcal{Z}^{Q^j} \simeq \frac{Z}{V^{3/2} \det |\sigma|} \left[\prod_{j=1}^3 \int_{-\infty}^{\infty} \frac{d\theta^j}{2\pi} \right] \exp \left[-i \xi^j \theta_j - \frac{\theta^j \theta_j}{2!} + \sum_{n=3}^{\infty} i^n V^{-\frac{n}{2}+1} \frac{\lambda_n^{j_1, j_2, \dots, j_n}}{n!} \theta_{j_1} \theta_{j_2} \dots \theta_{j_n} \right] . \quad (19)$$

Eq.(19) is the starting point for obtaining an asymptotic solution in this section as well as for finite volume corrections in Section VII. Through coordinate transformation Eq.(14) we have explicitly separated terms in their power in volume. Thus as system size is increased influence of higher order normalized cumulants λ_n decreases, allowing for truncation of the summation for sufficiently large volume. A few words on physical units are in order. The single particle partition function Eq.(8) $\psi_l[fm^{-3}]$, and therefore all cumulant elements Eq.(10) $\kappa[fm^{-3}]$ in CE, consequently entries in Eq.(15) $\sigma[fm^{-3/2}]$. The normalization in Eq.(19) $V^{3/2} \det |\sigma|$ for 3-dim σ , as well as the new variable of integration Eq.(14) θ^j , are hence dimensionless. The inverse sigma tensor elements are $\sigma^{-1}[fm^{3/2}]$ and thus the charge vector Eq.(18) ξ^j will be dimensionless. Finally normalized cumulants Eq.(17) are $\lambda_n[fm^{-3+3n/2}]$, which is canceled by the factor $V^{-n/2+1}$ in the summation in Eq.(19). Thus all terms involved in Eq.(19) are dimensionless.

For $V \rightarrow \infty$ one can discard terms of $V^{-1/2}$ and higher:

$$\mathcal{Z}^{Q^j} \simeq \frac{Z}{V^{3/2} \det |\sigma|} \left[\prod_{j=1}^3 \int_{-\infty}^{\infty} \frac{d\theta^j}{2\pi} \right] \exp \left[-i \xi^j \theta_j - \frac{\theta^j \theta_j}{2!} \right] . \quad (20)$$

Completing the square, the integral (20) can be solved:

$$\mathcal{Z}^{Q^j} \simeq Z \frac{\exp \left(-\frac{\xi^j \xi_j}{2} \right)}{(2\pi V)^{3/2} \det |\sigma|} . \quad (21)$$

From Section II it follows that in GCE one would find for the 3-dim charge PDF:

$$P(Q^j) = \frac{e^{\frac{Q^j \mu_j}{T}} Z^{Q^j}}{Z} = \frac{\mathcal{Z}^{Q^j}}{Z} \simeq \frac{\exp\left(-\frac{\xi^j \xi_j}{2}\right)}{(2\pi V)^{3/2} \det|\sigma|}. \quad (22)$$

Please note that Eq.(22) (albeit in different notation) was used as an assumption in the microscopic correlator approach [24, 25, 26, 27, 28].

B. Particle Number Distribution

Similarly to Eq.(6) the canonical state involving a charge vector Q^j and a particular number of particles N of some species can be described by the following 4-dim partition function:

$$\begin{aligned} \mathcal{Z}^{N,Q,B,S} = \int_{-\pi}^{\pi} \frac{d\phi_N}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi_Q}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi_B}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi_S}{2\pi} e^{-iN\phi_N} e^{-iQ\phi_Q} e^{-iB\phi_B} e^{-iS\phi_S} \\ \times \exp \left[\sum_{l=1} z_l (\phi_N, \phi_Q, \phi_B, \phi_S) \right], \end{aligned} \quad (23)$$

where $(N, Q, B, S) = (N, Q^j) = \tilde{Q}^j$. Each particle specie which is selected receives a Wick rotated fugacity $\exp(i\phi_N)$ in complete analogy to those associated with charge conservation. Similar to Eq.(21) the canonical partition function $\mathcal{Z}^{\tilde{Q}^j}$ can be approximated by:

$$\mathcal{Z}^{\tilde{Q}^j} \simeq Z \frac{\exp\left(-\frac{\tilde{\xi}^j \tilde{\xi}_j}{2}\right)}{(2\pi V)^{4/2} \det|\tilde{\sigma}|}, \quad (24)$$

Therefore in GCE we find the 4-dim PDF of finding our system in a state \tilde{Q}^j :

$$P(\tilde{Q}^j) = \frac{\mathcal{Z}^{\tilde{Q}^j}}{Z} \simeq \frac{\exp\left(-\frac{\tilde{\xi}^j \tilde{\xi}_j}{2}\right)}{(2\pi V)^{4/2} \det|\tilde{\sigma}|}. \quad (25)$$

However we are only interested in a 1-dim slice with constant values of conserved charges $Q^j = (Q, B, S)$, e.g. the conditional PDF $P(N|Q^j) = \mathcal{Z}^{\tilde{Q}^j} / \mathcal{Z}^{Q^j}$, where the normalization is Eq.(21) from the previous section. The expansion works best around the peak of the distribution. If analytical solutions were available this would not matter much, but here we need to find the peak of the PDF. In the thermodynamic limit peak and mean of the distribution coincide (see Section VII), hence it peaks at $Q_{eq}^k = V\kappa_1^k = (\langle Q \rangle, \langle B \rangle, \langle S \rangle)$. For the 3-dim distribution Eq.(21) from the previous section it follows that Eq.(18) $\xi^j = (Q_{eq}^k - V\kappa_1^k) (\sigma^{-1})_k^j V^{-1/2} = 0$. For the 4-dim

distribution of this section we find at $\tilde{Q}_{eq}^k = (N, Q_{eq}^k)$ for $\tilde{\xi}^j \tilde{\xi}_j = (N - \langle N \rangle)^2 V^{-1} (\tilde{\sigma}^{-1})_1^j (\tilde{\sigma}^{-1})_j^1$, and $\langle N \rangle = V \kappa_1^N$. The canonical PDF can thus be written as:

$$P(N|Q_{eq}^j) \simeq \frac{\det |\sigma|}{(2\pi V)^{1/2} \det |\tilde{\sigma}|} \exp \left(- (N - \langle N \rangle)^2 \frac{(\tilde{\sigma}^{-1})_1^j (\tilde{\sigma}^{-1})_j^1}{2V} \right). \quad (26)$$

Hence one can find the width of the distribution from its normalization, as well as from its exponential $\det |\sigma| / \det |\tilde{\sigma}| = \sqrt{(\tilde{\sigma}^{-1})_1^j (\tilde{\sigma}^{-1})_j^1} = (\langle N^2 \rangle - \langle N \rangle^2)^{-1/2} V^{1/2}$. The identity

$$\frac{\det |\kappa_2|}{\det |\tilde{\kappa}_2|} = (\tilde{\sigma}^{-1})_1^j (\tilde{\sigma}^{-1})_j^1. \quad (27)$$

is proven in Appendix (D). One gets for the scaled variance of the particle number distribution:

$$\omega = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} = \frac{V \det |\tilde{\sigma}|^2}{V \kappa_1^N \det |\sigma|^2} = \frac{\det |\tilde{\kappa}_2|}{\kappa_1^N \det |\kappa_2|}. \quad (28)$$

In words, this is the ratio of the product of the eigenvalues of the 3+1 dimensional matrix $\tilde{\kappa}_2$ and the 3 dimensional matrix κ_2 divided by the particle density of the particle under investigation. The equivalence of Eq.(28) to results of the micro-correlator approach, see e.g. [24, 25, 26, 27, 28], will be shown in Section VI.

$$P(N|Q_{eq}^j) \simeq \frac{1}{(2\pi\omega\langle N \rangle)^{1/2}} \exp \left(- \frac{(N - \langle N \rangle)^2}{2\omega\langle N \rangle} \right). \quad (29)$$

Thus we obtained the rigorous mathematical proof of the hypothesis that the multiplicity distributions in a hadron-resonance gas in thermodynamic limit have Gaussian shape. Diagonalization of κ_2 was not at all necessary for arriving at the asymptotic solution Eq.(28). However for finite volume corrections one will have to go through the process of finding eigenvectors and eigenvalues of κ_2 (see Section VII).

C. The Cumulant Tensor

In this section we will calculate the κ -tensor Eq.(10) only for primordial single particle specie fluctuations. In Section V this will be extended to fluctuations of a selection of particles, like ‘positively charged’ and resonance decay will be included.

$$\kappa_1^{j_1} = \left(-i \frac{\partial}{\partial \phi_{j_1}} \right) \Psi \Big|_{\phi^j = \vec{0}}, \quad (30)$$

and

$$\kappa_2^{j_1, j_2} = \left(-i \frac{\partial}{\partial \phi_{j_1}} \right) \left(-i \frac{\partial}{\partial \phi_{j_2}} \right) \Psi \Big|_{\phi^j = \vec{0}} . \quad (31)$$

Cumulants of order 1 give GCE expectation values, hence average baryon, strangeness, electric charge, and particle density. The second cumulant contains information about GCE fluctuations of some quantity (diagonal elements), as well as correlations between different quantities (off-diagonal elements). For the distributions $P(Q^j)$ and $P(\tilde{Q}^j)$ the first and second cumulants are:

$$\kappa_1 = \left(\kappa_1^Q, \kappa_1^B, \kappa_1^S \right) , \quad \tilde{\kappa}_1 = \left(\kappa_1^N, \kappa_1^Q, \kappa_1^B, \kappa_1^S \right) , \quad (32)$$

$$\kappa_2 = \begin{pmatrix} \kappa_2^{Q,Q} & \kappa_2^{Q,B} & \kappa_2^{Q,S} \\ \kappa_2^{B,Q} & \kappa_2^{B,B} & \kappa_2^{B,S} \\ \kappa_2^{S,Q} & \kappa_2^{S,B} & \kappa_2^{S,S} \end{pmatrix} , \quad \tilde{\kappa}_2 = \begin{pmatrix} \kappa_2^{N,N} & \kappa_2^{N,Q} & \kappa_2^{N,B} & \kappa_2^{N,S} \\ \kappa_2^{Q,N} & \kappa_2^{Q,Q} & \kappa_2^{Q,B} & \kappa_2^{Q,S} \\ \kappa_2^{B,N} & \kappa_2^{B,Q} & \kappa_2^{B,B} & \kappa_2^{B,S} \\ \kappa_2^{S,N} & \kappa_2^{S,Q} & \kappa_2^{S,B} & \kappa_2^{S,S} \end{pmatrix} . \quad (33)$$

In case two quantities are un-correlated, as for example primordial π^+ multiplicity and globally conserved strangeness (π^+ does not carry strangeness), then the corresponding elements $\kappa_2^{N,S} = \kappa_2^{S,N} = 0$. For clarity some elements are explicitly given. The primordial mean value of particle number density of particle species l and the mean charge density are:

$$\kappa_1^N = \left(-i \frac{\partial}{\partial \phi_N} \right) \Psi \Big|_{\phi_j = \vec{0}} = \psi'_l , \quad (34)$$

$$\kappa_1^Q = \left(-i \frac{\partial}{\partial \phi_Q} \right) \Psi \Big|_{\phi_j = \vec{0}} = \sum_l q_l \psi'_l . \quad (35)$$

Fluctuation of particle density $\kappa_2^{N,N}$, correlation between particle number and baryonic charge $\kappa_2^{N,B}$, and correlation between strangeness and baryonic charge $\kappa_2^{S,B}$ are given by:

$$\kappa_2^{N,N} = \left(-i \frac{\partial}{\partial \phi_N} \right)^2 \Psi \Big|_{\phi_j = \vec{0}} = \psi''_l , \quad (36)$$

$$\kappa_2^{N,B} = \left(-i \frac{\partial}{\partial \phi_N} \right) \left(-i \frac{\partial}{\partial \phi_B} \right) \Psi \Big|_{\phi_j = \vec{0}} = b_l \psi''_l , \quad (37)$$

$$\kappa_2^{S,B} = \left(-i \frac{\partial}{\partial \phi_S} \right) \left(-i \frac{\partial}{\partial \phi_B} \right) \Psi \Big|_{\phi_j = \vec{0}} = \sum_l s_l b_l \psi''_l . \quad (38)$$

where the first and second derivative of ψ are:

$$\psi'_l = \frac{g_l}{(2\pi)^3} \int d^3p \frac{e^{-(\varepsilon_l - \mu_l)/T}}{(1 \pm e^{-(\varepsilon_l - \mu_l)/T})} , \quad (39)$$

$$\psi''_l = \frac{g_l}{(2\pi)^3} \int d^3p \frac{e^{-(\varepsilon_l - \mu_l)/T}}{(1 \pm e^{-(\varepsilon_l - \mu_l)/T})^2} . \quad (40)$$

Please note that the analogs of the matrix κ_2 and the vector κ_1 , however in different notation, were used in both previously published methods for calculation of scaled variance under the thermodynamic limit, the micro-correlator approach, see e.g. [24, 25, 26, 27, 28], and saddle point expansion method [29]. The advantage of this paper is certainly the very simple formula (28), and the possibility for calculation of finite volume corrections (see Section VII), giving a lower bound for the validity of the micro-correlator approach and saddle point expansion. Exact agreement was found with the analytical results for the scaled variance in the thermodynamic limit [19, 24, 25, 26, 27, 28]. Numerical calculations give very good agreement to [29], despite the fact that different particle tables were used in the latter.

IV. MICROCANONICAL ENSEMBLE

In the MCE we additionally enforce kinematic conservation laws. The simplest example is an ultra-relativistic gas, made up of one kind of neutral massless particles. The reason for this choice is that an analytical solution exists, at least for a microcanonical ensemble, where energy but not momentum is conserved [1, 27], in MB approximation, allowing for a comparison of analytic results to asymptotic solutions (see Sections VII C and VIII C). Further we can carry out the integration of the GGPF in full detail for this example (Appendix C), and highlight some features of our method, such as the role of temperature in the MCE.

In this section, however, we explicitly include momentum conservation as well as FD and BE statistics, and restrict ourself to the asymptotic solution. The basic ideas are essentially the same as before. The only conserved quantities here then are the total energy E and the three momentum \vec{P} . In principle we would have to treat energy and momentum conservation on equal footing and introduce Lagrange multipliers associated with conserved momenta. However, in the rest frame of a static thermal source we always find $\langle \vec{P} \rangle = \vec{0}$. Thus we only need to consider the 0^{th} component, $1/T$, of the ‘four-temperature’ [32]. The probability to find a microcanonical system in a state with exactly N particles is defined by:

$$P(N|E, \vec{P} = 0) = \frac{\text{number of all states with } N \text{ particles, } E, \text{ and } \vec{P} = 0}{\text{number of all states with } E, \text{ and } \vec{P} = 0} . \quad (41)$$

Our starting point is again the GCE partition function, which can be written as:

$$Z = \exp \left[V \frac{g}{(2\pi)^3} \int d^3p f(\vec{p}) \right] , \quad (42)$$

where $f(\vec{p})$ is the probability of having a particular momentum state occupied. For massless particles we find $\varepsilon = |\vec{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2}$, thus $f(\vec{p}) = \exp(-|\vec{p}|/T)$ in MB approximation, or in the quantum statistical treatment $f(\vec{p}) = \ln(1 \pm \exp(-|\vec{p}|/T))^{\pm 1}$, where the upper sign denotes FD, while the lower sign stands for BE statistics. The numerator in Eq.(41) is given by the 5-dim Fourier integral over the GGPF $\mathcal{Z}(\phi_N, \phi_E, \vec{\phi}_p)$:

$$P(N, E, \vec{P}) = \frac{1}{Z} \int_{-\pi}^{\pi} \frac{d\phi_N}{2\pi} \int_{-\infty}^{\infty} \frac{d\phi_E}{2\pi} \int_{-\infty}^{\infty} \frac{d\vec{\phi}_p}{(2\pi)^3} e^{-iN\phi_N} e^{-iE\phi_E} e^{-i\vec{P}\vec{\phi}_p} \exp[V\Psi(\phi_N, \phi_E, \vec{\phi}_p)] . \quad (43)$$

Energy and momentum conservation (continuous quantities) require additionally the use of Dirac δ functions, rather than the Kronecker δ for discrete quantities. We need dimensionless quantities for the delta-functions, thus ϕ_E and $\vec{\phi}_p$ need to have dimension $[GeV^{-1}]$ (see as well Section VIII C, Appendix C, and ref. [32]). The CGF can be expressed with a suitable choice for ϕ_E and $\vec{\phi}_p$ in MB statistics as:

$$\Psi(\phi_N, \phi_E, \vec{\phi}_p) = \frac{g}{(2\pi)^3} \int d^3p e^{-|\vec{p}|/T} e^{i\phi_N} e^{i|\vec{p}|\phi_E} e^{i\vec{p}\vec{\phi}_p} , \quad (44)$$

or for FD (upper sign) and BE (lower sign) statistics as:

$$\Psi(\phi_N, \phi_E, \vec{\phi}_p) = \frac{g}{(2\pi)^3} \int d^3p \ln \left[1 \pm e^{-|\vec{p}|/T} e^{i\phi_N} e^{i|\vec{p}|\phi_E} e^{i\vec{p}\vec{\phi}_p} \right]^{\pm 1} . \quad (45)$$

Using shorthand $\tilde{Q}^j = (N, E, P_x, P_y, P_z)$ and $\tilde{\phi}_j = (\phi_N, \phi_E, \phi_{p_x}, \phi_{p_y}, \phi_{p_z})$ one can simplify the notation, and only use the asymptotic solution Eq.(22) for large volumes, derived in the previous section.

$$\begin{aligned} P(\tilde{Q}^j) &\simeq \left[\prod_{j=1}^5 \int_{-\infty}^{\infty} \frac{d\tilde{\phi}_j}{(2\pi)} \right] e^{-i\tilde{Q}^j \tilde{\phi}_j} \exp \left(V \sum_{n=1}^{\infty} \frac{i^n}{n!} \tilde{\kappa}_n^{j_1 \dots j_n} \phi_{j_1} \dots \phi_{j_n} \right) \\ &\simeq \frac{1}{(2\pi V)^{5/2}} \frac{\exp\left(-\frac{\tilde{\xi}^j \tilde{\xi}_j}{2}\right)}{\det |\tilde{\sigma}|} . \end{aligned} \quad (46)$$

The cumulants needed are now given by the respective derivatives at the origin. In case FD or BE statistics are used one will employ the same set of derivative operators, which result in the

usual, yet slightly more difficult, integrals due to the logarithm. The expectation values are:

$$\kappa_1^N = \left(-i \frac{\partial}{\partial \phi_N} \right) \Psi \Big|_{\phi_j=\vec{0}} = \frac{g}{(2\pi)^3} \int d^3p f'(\vec{p}) , \quad (47)$$

$$\kappa_1^E = \left(-i \frac{\partial}{\partial \phi_E} \right) \Psi \Big|_{\phi_j=\vec{0}} = \frac{g}{(2\pi)^3} \int d^3p f'(\vec{p}) |\vec{p}| , \quad (48)$$

$$\kappa_1^{p_x} = \left(-i \frac{\partial}{\partial \phi_{p_x}} \right) \Psi \Big|_{\phi_j=\vec{0}} = \frac{g}{(2\pi)^3} \int d^3p f'(\vec{p}) p_x = 0 , \quad (49)$$

while some selected elements from the second rank tensor $\tilde{\kappa}_2$ are:

$$\kappa_2^{N,N} = \left(-i \frac{\partial}{\partial \phi_N} \right)^2 \Psi \Big|_{\phi_j=\vec{0}} = \frac{g}{(2\pi)^3} \int d^3p f''(\vec{p}) , \quad (50)$$

$$\kappa_2^{E,E} = \left(-i \frac{\partial}{\partial \phi_E} \right)^2 \Psi \Big|_{\phi_j=\vec{0}} = \frac{g}{(2\pi)^3} \int d^3p f''(\vec{p}) |\vec{p}|^2 , \quad (51)$$

$$\kappa_2^{p_x,p_x} = \left(-i \frac{\partial}{\partial \phi_{p_x}} \right)^2 \Psi \Big|_{\phi_j=\vec{0}} = \frac{g}{(2\pi)^3} \int d^3p f''(\vec{p}) p_x^2 = \frac{1}{3} \kappa_2^{E,E} , \quad (52)$$

$$\kappa_2^{E,N} = \left(-i \frac{\partial}{\partial \phi_E} \right) \left(-i \frac{\partial}{\partial \phi_N} \right) \Psi \Big|_{\phi_j=\vec{0}} = \frac{g}{(2\pi)^3} \int d^3p f''(\vec{p}) |\vec{p}| , \quad (53)$$

where similar to Eqs. (39) and (40) we define:

$$f'(\vec{p}) = \frac{e^{-|\vec{p}|/T}}{1 \pm e^{-|\vec{p}|/T}} , \quad \text{and} \quad f''(\vec{p}) = \frac{e^{-|\vec{p}|/T}}{(1 \pm e^{-|\vec{p}|/T})^2} , \quad (54)$$

or simply $f(\vec{p}) = f'(\vec{p}) = f''(\vec{p})$ in MB approximation. Due to spherical symmetry of the momentum distribution, we find $\kappa_2^{p_x,p_x} = \kappa_2^{p_y,p_y} = \kappa_2^{p_z,p_z}$. Further correlation terms are identical to zero, $\kappa_2^{N,p_x} = \kappa_2^{E,p_x} = \kappa_2^{p_x,p_y} = 0$. In our previous notation we find the first and second cumulants, $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$, in Boltzmann approximation for the distribution $P(N, E, \vec{P})$:

$$\tilde{\kappa}_1 = \frac{g}{(2\pi)^3} \left(8\pi T^3, 24\pi T^4, 0, 0, 0 \right) , \quad \text{and} \quad (55)$$

$$\tilde{\kappa}_2 = \frac{g}{(2\pi)^3} \begin{pmatrix} 8\pi T^3 & 24\pi T^4 & 0 & 0 & 0 \\ 24\pi T^4 & 96\pi T^5 & 0 & 0 & 0 \\ 0 & 0 & 32\pi T^5 & 0 & 0 \\ 0 & 0 & 0 & 32\pi T^5 & 0 \\ 0 & 0 & 0 & 0 & 32\pi T^5 \end{pmatrix} . \quad (56)$$

For the normalization we need additionally the second cumulant κ_2 of the PDF $P(E, \vec{P})$, which is obtained from $\tilde{\kappa}_2$ by crossing out the 1st column and 1st row. The remaining relevant integrals

	MB	FD	BE
κ_1^N	$8\pi T^3$	$6\pi\zeta(3)T^3$	$8\pi\zeta(3)T^3$
κ_1^E	$24\pi T^4$	$\frac{7}{30}\pi^5 T^4$	$\frac{4}{15}\pi^5 T^4$
$\kappa_2^{N,N}$	$8\pi T^3$	$\frac{2}{3}\pi^3 T^3$	$\frac{4}{3}\pi^3 T^3$
$\kappa_2^{E,N}$	$24\pi T^4$	$18\pi\zeta(3)T^4$	$24\pi\zeta(3)T^4$
$\kappa_2^{E,E}$	$96\pi T^5$	$\frac{14}{15}\pi^5 T^5$	$\frac{16}{15}\pi^5 T^5$

TABLE I: Selected elements of the first two cumulant tensors for a massless gas in Fermi-Dirac, Bose-Einstein statistics and Boltzmann approximation. All entries have to be multiplied by $g/(2\pi)^3$. The Riemann Zeta function is $\zeta(3) \simeq 1.202$.

for FD and BE statistics are summarized in table I. Following the recipe from above, see Eq.(28), the scaled variance can be expressed as follows:

$$\omega = \frac{\det |\tilde{\kappa}_2|}{\kappa_1^N \det |\kappa_2|} = \frac{\kappa_2^{E,E} \kappa_2^{N,N} - \left(\kappa_2^{E,N}\right)^2}{\kappa_1^N \kappa_2^{E,E}}. \quad (57)$$

Independent of energy density one finds for the large volume limit the following asymptotic values for the scaled variance ω , in exact agreement with [27], in table II. Momentum conser-

	MB	FD	BE
ω_{mce}	0.25	0.198314	0.535463

TABLE II: Asymptotic of the scaled variance in a MCE for a neutral massless gas

vation explicitly drops out in the thermodynamic limit in the calculation of the ratio of the relevant determinants. This will not hold true for any finite system size (see Section VII), due to the appearance of off-diagonal terms symmetric in momentum like κ_3^{N,p_x,p_x} . However for large enough volumes, expected to be created in heavy ion collision experiments, one can probably safely disregard exact momentum conservation, when considering multiplicity fluctuations in the full momentum space, i.e. 4π yield and fluctuations. In previous publications on asymptotic multiplicity fluctuations in the MCE it was only always assumed (without proof) that exact momentum conservation would not affect the result, and thus for technical reasons not taken into account.

One last issue should be mentioned. For partition functions for very small volume one should return to summation over (quantized) momentum states, rather than integration over (continuous) momentum space, or ideally turn to a quantum field theoretical frame work of the MCE [11]. Usually these distinctions do not affect the result much, simplify however calculations considerably. For a massless gas however we encounter the problem of a divergent $\kappa_3^{N,N,N}$ for BE statistics. This problem is well known from textbooks and usually overcome by applying a low momentum cut-off [33].

V. RESONANCE DECAY

In this section we want to show how to include resonance decay analytically directly into the system partition function. This will prove far more efficient than the definition of a generating function [26] which requires a rather cumbersome calculation of all possible primordial correlators, see Section VIB. Particle decay is itself a random process. Nevertheless one can assign a particular volume in phase space, given by the value of its single particle partition function z_l , Eq.(8), to one type of resonance l . Resonance decay will now populate this volume in phase space according to some weight factor, the branching ratio, for each of the possible decay modes. This weight can be assigned to the particle type(s) we are set to investigate. Based on the assumption that detected particles are drawn in the form of a random sample from all final state particles, e.g. disregarding correlation in momentum space, this procedure leads to acceptance scaling employed in ref. [19, 23, 26] (see Section VIC). Conservation laws can be imposed on the primordial state (rather than the final state), since decay channels, which are experimentally measured, do not only obey charge conservation, but all relevant conservation laws (omitting weak decays).

A. Final State Partition Function

For calculation of final state distributions, we have to determine *final* branching ratios of a resonance type into only stable particles. As an example we consider the decay channel $A \rightarrow B + X$ with branching ratio $\Gamma_{A \rightarrow B+X} = a$. Resonance B could itself be unstable and subsequently decay via the channel $B \rightarrow Y + Z$ with branching ratio $\Gamma_{B \rightarrow Y+Z} = b$. So we define the *final* branching ratio $\Gamma_{A \rightarrow X+Y+Z} = \Gamma_{A \rightarrow B+X} \Gamma_{B \rightarrow Y+Z} = ab$. Decay tables in [19, 26]

have also been generated according to this prescription.

For resonances it seems economical to define further *absolute* branching ratios Γ_l^c as the sum over all *final* decay channels of resonance l with a given number c of selected daughters. Hence Γ_l^2 is the sum over all *final* decay channels with two daughter particles which are of interest, i.e. two positively charged particles in case one wants to calculate ω^+ . As a consequence of this definition, branching ratios Γ_l^c will depend on which ω one is set to calculate.

For the final state one has to take all C_l *absolute* decay channels of resonance type l into a number c of selected stable particles into account. All parent resonances carry fictitious particle specific fugacity λ_N taken to the power of the number c of selected daughters in a particular channel, $\Gamma_l^c (\lambda_N)^c$. For the sake of a common treatment for all particles and resonances we assign a ‘decay’ channel to stable particles as well, either $\Gamma_l^1 = 1$ (while $\Gamma_l^{c \neq 1} = 0$) if selected, or $\Gamma_l^0 = 1$ (while $\Gamma_l^{c \neq 0} = 0$) if not selected. The single particle partition function reads after substitution $(\lambda_N)^c \rightarrow e^{ic\phi_N}$:

$$\psi_l(\phi_j, \phi_N, \Gamma_l^c) = \frac{g_l}{(2\pi)^3} \int d^3p \ln \left(1 \pm e^{-(\varepsilon_l - \mu_l)/T} e^{iq_l^j \phi_j} \left[\sum_{c=0}^{C_l} \Gamma_l^c e^{ic\phi_N} \right] \right)^{\pm 1}, \quad (58)$$

where the form of the vector q_l^j depends on our choice of ensemble. For instance, we could have for a hadron resonance gas with three conserved charges $q_l^j = (q_l, b_l, s_l)$ in the CE, or $q_l^j = (q_l, b_l, s_l, \varepsilon_l, \vec{p}_l)$ in the MCE. The sum over all decay channels which produce from zero up to a number C_l particles of the selected types needs to be one:

$$\sum_{c=0}^{C_l} \Gamma_l^c = 1. \quad (59)$$

This is somewhat of a practical challenge, since decay chains of heavier resonances are not always well established [35] and respective thermal models codes [36, 37] struggle to implement this. There are several ways to deal with this, the two extreme ones are 1) rescale all known channels according to Eq.(59), to unity, or 2) assign the missing fraction to the ‘channel’ Γ_l^0 , e.g. to the channel without stable particles of interest. In case one sets all angles $(\phi_j, \phi_N) = \vec{0}$, i.e. one returns to $e^{ic\phi_N} \rightarrow (\lambda_N)^c$, one obtains the GCE partition function:

$$Z = \exp \left[V \sum_l \frac{g_l}{(2\pi)^3} \int d^3p \ln \left(1 \pm e^{-(\varepsilon_l - \mu_l)/T} \left[\sum_{c=0}^{C_l} \Gamma_l^c (\lambda_N)^c \right] \right)^{\pm 1} \right], \quad (60)$$

from which GCE expectation values can be calculated as $\langle N \rangle = \lambda_N \frac{\partial}{\partial \lambda_N} \ln Z|_{\lambda_N=1}$.

B. Limited Acceptance

An imperfect detector will modify our definition of branching ratio Γ_l^c . Based on the assumption of un-correlated particle detection, all particles have equal probability q of being observed. The corresponding acceptance distribution of observing n particles when c are produced is given by a binomial distribution $P_{acc}(n, c) = q^n (1 - q)^{c-n} \binom{c}{n}$. We define the *effective* branching ratios $\Gamma_{l,q}^n$, which already include the effect of finite acceptance, as the product of *absolute* branching ratio Γ_l^c and acceptance distribution:

$$\Gamma_{l,q}^n = \sum_{c=n}^{C_l} \Gamma_l^c q^n (1 - q)^{c-n} \binom{c}{n}. \quad (61)$$

For example the *effective* branching ratio $\Gamma_{l,q}^2$ is the sum over all *absolute* branching ratios Γ_l^c which produce at least 2 stable selected particle, $\Gamma_{l,q}^2 = \Gamma_l^2 q^2 + 3\Gamma_l^3 q^2 (1 - q) + 6\Gamma_l^4 q^2 (1 - q)^2 + \dots$. The binomial coefficient $\binom{c}{n} = \frac{c!}{n!(c-n)!}$ takes care of the fact that particles are indistinguishable. In our approximation resonance decay and particle detection are thus two independent random processes. When both, particle decay and un-correlated detection, are included, the single particle partition functions reads:

$$\psi_{l,q}(\phi_j, \phi_N, \Gamma_l^c, q) = \frac{g_l}{(2\pi)^3} \int d^3p \ln \left(1 \pm e^{-(\varepsilon_l - \mu_l)/T} e^{iq_l^j \phi_j} \left[\sum_{n=0}^{C_l} \Gamma_{l,q}^n e^{in\phi_N} \right] \right)^{\pm 1}. \quad (62)$$

The sum over all *effective* channels $\Gamma_{l,q}^n$ is equal to one,

$$\sum_{n=0}^{C_l} \Gamma_{l,q}^n = \sum_{c=0}^{C_l} \Gamma_l^c \sum_{n=0}^c q^n (1 - q)^{c-n} \binom{c}{n} = 1, \quad (63)$$

since the second summation ($\sum_{n=0}^c$) is equal to unity, while according to Eq.(59) the branching ratios Γ_l^c are also normalized. The final state CGF thus is the sum over all such single particle partition functions:

$$\Psi = \sum_l \psi_{l,q}(\phi_j, \phi_N, \Gamma_{l,q}^c). \quad (64)$$

C. The Cumulant Tensor

The procedure is essentially no different from Sections III and IV, yet slightly more complicated, due to the various multiplicities in the decay modes. For clarity some elements are

explicitly given. The choice of ensemble then naturally defines the cumulants needed for calculations. Both, CE and MCE, are considered. The final mean values of particle and energy density are:

$$\kappa_1^N = \left(-i \frac{\partial}{\partial \phi_N} \right) \Psi \Big|_{\phi_j=\vec{0}} = \sum_l \left[\sum_{n=0}^{C_l} \Gamma_{l,q}^n n \right] \psi_l^{(1,1;0)} , \quad (65)$$

$$\kappa_1^E = \left(-i \frac{\partial}{\partial \phi_E} \right) \Psi \Big|_{\phi_j=\vec{0}} = \sum_l \psi_l^{(1,1;1)} , \quad (66)$$

Fluctuation of particle and energy density, and correlations between particle and baryon number, particle number and energy, and between baryon number and energy are given by:

$$\kappa_2^{N,N} = \left(-i \frac{\partial}{\partial \phi_N} \right)^2 \Psi \Big|_{\phi_j=\vec{0}} = \sum_l \left[\sum_{n=0}^{C_l} \Gamma_{l,q}^n n^2 \right] \psi_l^{(1,1;0)} - \left[\sum_{n=0}^{C_l} \Gamma_{l,q}^n n \right]^2 \psi_l^{(2,2;0)} , \quad (67)$$

$$\kappa_2^{E,E} = \left(-i \frac{\partial}{\partial \phi_E} \right)^2 \Psi \Big|_{\phi_j=\vec{0}} = \sum_l \psi_l^{(1,2;2)} , \quad (68)$$

$$\kappa_2^{N,B} = \left(-i \frac{\partial}{\partial \phi_B} \right) \left(-i \frac{\partial}{\partial \phi_N} \right) \Psi \Big|_{\phi_j=\vec{0}} = \sum_l b_l \left[\sum_{n=0}^{C_l} \Gamma_{l,q}^n n \right] \psi_l^{(1,2;0)} , \quad (69)$$

$$\kappa_2^{N,E} = \left(-i \frac{\partial}{\partial \phi_E} \right) \left(-i \frac{\partial}{\partial \phi_N} \right) \Psi \Big|_{\phi_j=\vec{0}} = \sum_l \left[\sum_{n=0}^{C_l} \Gamma_{l,q}^n n \right] \psi_l^{(1,2;1)} , \quad (70)$$

$$\kappa_2^{B,E} = \left(-i \frac{\partial}{\partial \phi_E} \right) \left(-i \frac{\partial}{\partial \phi_B} \right) \Psi \Big|_{\phi_j=\vec{0}} = \sum_l b_l \psi_l^{(1,2;1)} , \quad (71)$$

where we use a more general shorthand notation for the derivatives similar to Eqs.(39), and (40):

$$\psi_l^{(a,b;c)} = (\pm 1)^{a+1} \frac{g_l}{(2\pi)^3} \int d^3p \, \varepsilon_l^c \frac{(e^{-(\varepsilon_l - \mu_l)/T})^a}{(1 \pm e^{-(\varepsilon_l - \mu_l)/T})^b} . \quad (72)$$

It is quite easy to see that higher orders of these elements become quickly very complicated. For calculation of asymptotic multiplicity fluctuations, integrated over full momentum space (4π fluctuations), it is not necessary to take momentum conservation into account (see Section IV). The implementation of proper kinematical cuts will be the subject of a forthcoming paper.

VI. CORRESPONDENCE TO MICROSCOPIC CORRELATOR APPROACH

One can make a correspondence between different notations in this paper and in the micro-correlator approach published previously [19, 24, 25, 26, 27, 28] for asymptotic fluctuations.

The aim is to show that both methods lead to the same result, as well as handle resonance decay exactly the same way. This fact should not be taken as trivial, as both methods appear very differently (not only in terms of notation). In this work all quantities are derived from a macroscopic partition function, while in the microscopic correlator approach one considers average occupation numbers, and their fluctuations, of individual momentum levels and correlations between different levels. We restrict the consideration to CE with three conserved charges. Additional charges or energy (momentum) conservation in the MCE would just lead to enlargement of the matrixes κ_2 and $\tilde{\kappa}_2$. We start off from Eq.(28):

$$\omega_{c.e.} = \frac{\det |\tilde{\kappa}_2|}{\kappa_1^N \det |\kappa_2|}, \quad (73)$$

where $\det |\kappa_2|$ and $\det |\tilde{\kappa}_2|$ are the determinants of the matrices Eq.(33). Expanding the determinants in terms of their complementary minors, $M_{i,k}$ and $\widetilde{M}_{i,k}$ respectively, yields [38]:

$$\det |\tilde{\kappa}_2| = \sum_{j=1}^4 (-1)^{j+k} (\tilde{\kappa}_2)_{j,k} \widetilde{M}_{j,k}, \quad (74)$$

where minors of $\tilde{\kappa}_2$ can be expressed in terms of the minors of κ_2 :

$$\widetilde{M}_{N,N} = \det |\kappa_2|, \quad \widetilde{M}_{N,Q} = \kappa_2^{Q,N} M_{Q,Q} - \kappa_2^{B,N} M_{B,Q} + \kappa_2^{S,N} M_{S,Q}, \quad \text{etc.} \quad (75)$$

After straightforward calculation Eq. (73) can be rewritten as follows:

$$\begin{aligned} \omega_{c.e.} &= \frac{\kappa_2^{N,N} \widetilde{M}_{N,N} - \kappa_2^{N,Q} \widetilde{M}_{N,Q} + \kappa_2^{N,B} \widetilde{M}_{N,B} + \kappa_2^{N,S} \widetilde{M}_{N,S}}{\kappa_1^N \det |\kappa_2|} \\ &= \frac{\kappa_2^{N,N}}{\kappa_1^N} - \frac{1}{\kappa_1^N \det |\kappa_2|} \left[\left(\kappa_2^{N,Q} \right)^2 M_{Q,Q} + \left(\kappa_2^{N,B} \right)^2 M_{B,B} + \left(\kappa_2^{N,S} \right)^2 M_{S,S} \right. \\ &\quad \left. + 2\kappa_2^{N,Q} \kappa_2^{N,S} M_{Q,S} - 2\kappa_2^{N,Q} \kappa_2^{N,B} M_{Q,B} - 2\kappa_2^{N,B} \kappa_2^{N,S} M_{B,S} \right]. \end{aligned} \quad (76)$$

This is the most general case. For a specific calculation we need to specify the matrix elements.

A. Primordial

In order to change our notation here to that of the microscopic correlator approach [19, 26], we need the derivatives Eq.(72) of the single particle partition function ψ_i :

$$V\psi_i^{1,1;0} \equiv \sum_p \langle n_{p,i} \rangle, \quad V\psi_i^{1,2;0} \equiv \sum_p v_{p,i}^2, \quad \text{and} \quad \psi_i^{2,2;0} \equiv \psi_i^{1,1;0} - \psi_i^{1,2;0},$$

where the subscripts i and p denotes particle species and momentum level respectively. The matrix entries from Section III C are:

$$\begin{aligned} V\kappa_1^N &= \langle N \rangle \equiv \sum_{p,i} \langle n_{p,i} \rangle & V\kappa_2^{N,N} &= \Delta(n^2) \equiv \sum_{p,i} v_{p,i}^2 & V\kappa_2^{N,Q} &= \Delta(nq) \equiv \sum_{p,i} q_i v_{p,i}^2 \\ V\kappa_2^{Q,Q} &= \Delta(q^2) \equiv \sum_{p,i} q_i^2 v_{p,i}^2 & V\kappa_2^{Q,B} &= \Delta(qb) \equiv \sum_{p,i} q_i b_i v_{p,i}^2 & \text{etc.} \end{aligned} ,$$

where [33] $v_{p,i} = \langle n_{p,i} \rangle (1 \pm \langle n_{p,i} \rangle)$ (upper sign for FD, lower sign for BE). It is easy to see that our matrix κ_2 corresponds to the correlation matrix $A \equiv V\kappa_2$, see e.g. Eq.(12) in [26]. Substituting into Eq.(76) one finds Eqs.(50,51) from reference [26]:

$$\omega_{c.e.} \equiv \frac{1}{\langle N \rangle} \sum_{i,j} \langle \Delta N_i \Delta N_j \rangle_{c.e.} , \quad (77)$$

with the correlator $\langle \Delta N_i \Delta N_j \rangle_{c.e.}$ being Eqs.(11,15) from [26]:

$$\begin{aligned} \langle \Delta N_i \Delta N_j \rangle_{c.e.} &= \sum_{p,k} \left[v_{p,i}^2 \delta_{ij} \delta_{pk} - \frac{v_{p,i}^2 v_{k,j}^2}{|A|} \left[q_i q_j M_{Q,Q} + b_i b_j M_{B,B} + s_i s_j M_{S,S} \right. \right. \\ &\quad \left. \left. + (q_i s_j + q_j s_i) M_{Q,S} - (q_i b_j + q_j b_i) M_{Q,B} - (b_i s_j + b_j s_i) M_{B,S} \right] \right] . \end{aligned} \quad (78)$$

Eq. (77) can be simplified for the case of single-specie fluctuations:

$$\begin{aligned} \omega_{c.e.}^j &= \omega_{g.c.e.}^j \left[1 - \frac{\sum_k v_{k,j}^2}{|A|} \left(q_j^2 M_{Q,Q} + b_j^2 M_{B,B} + s_j^2 M_{S,S} \right. \right. \\ &\quad \left. \left. + 2q_j s_j M_{Q,S} - 2q_j b_j M_{Q,B} - 2b_j s_j M_{B,S} \right) \right] , \end{aligned} \quad (79)$$

which coincides with Eq.(16) from [26], where $\omega_{g.c.e.}^j \equiv \sum_p v_{p,j}^2 / \sum_p \langle n_{p,j} \rangle$.

B. Resonance Decay

In order to account for resonance decay we need the matrix elements stated in Section V C together with:

$$\sum_{n=0}^{C_l} \Gamma_l^n n^2 = \sum_i \sum_j \langle n_i n_j \rangle_l , \quad \text{and} \quad \left(\sum_{n=0}^{C_l} \Gamma_l^n n \right)^2 = \sum_i \sum_j \langle n_i \rangle_l \langle n_j \rangle_l , \quad (80)$$

where the summations are to be taken over all stable hadrons i and j which one is set to consider, and $n = \sum_i n_i$ is the total number of selected particles in this channel. The matrix element $V_{\kappa_2^{N,N}} = \Delta(n^2)$ then splits into summations over stable hadron i, j and resonances R ,

$$\Delta(n^2) = \sum_{i,p} v_{i,p}^2 + \sum_{R,p} v_{R,p}^2 \sum_{i,j} \langle n_i \rangle_R \langle n_j \rangle_R + \sum_{R,p} n_{R,p} \sum_{i,j} \langle \Delta n_i \Delta n_j \rangle_R, \quad (81)$$

while the correlation terms $V_{\kappa_2^{N,Q}} = \Delta(nq)$ are of the form

$$\Delta(nq) = \sum_{i,p} q_i v_{i,p}^2 + \sum_{R,p} q_R v_{R,p}^2 \sum_i \langle n_i \rangle_R. \quad (82)$$

Substituting into Eq.(76) gives Eq.(47) in reference [26]:

$$\begin{aligned} \langle \Delta N_i \Delta N_j \rangle_{c.e.} &= \langle \Delta N_i^* \Delta N_j^* \rangle_{c.e.} + \sum_R \langle N_R \rangle \langle \Delta n_i \Delta n_j \rangle_R + \sum_R \langle \Delta N_i^* \Delta N_R \rangle_{c.e.} \langle n_j \rangle_R \\ &+ \sum_R \langle \Delta N_j^* \Delta N_R \rangle_{c.e.} \langle n_i \rangle_R + \sum_{R,R'} \langle \Delta N_R \Delta N_{R'} \rangle_{c.e.} \langle n_i \rangle_R \langle n_j \rangle_{R'}. \end{aligned} \quad (83)$$

Correlation terms $\langle \Delta N_i^* \Delta N_R \rangle_{c.e.}$, and $\langle \Delta N_R \Delta N_{R'} \rangle_{c.e.}$ appear in the CE (and the MCE) due to products of $\Delta(nq) \Delta(ns)$, etc., in Eq.(76), and are absent in the GCE.

C. Acceptance Scaling

Starting off again from our approximation of un-correlated particle detection, from Section VB, we find for the first two moments [38] of the binomial distribution of detected particles produced by decay of resonance type l :

$$\langle n \rangle_l = \sum_{c=0}^{C_l} \Gamma_l^c \sum_{n=0}^c n q (1-q)^{c-n} \binom{c}{n} = \sum_{c=0}^{C_l} \Gamma_l^c q c = q \langle c \rangle_l, \quad (84)$$

$$\begin{aligned} \langle n^2 \rangle_l &= \sum_{c=0}^{C_l} \Gamma_l^c \sum_{n=0}^c n^2 q (1-q)^{c-n} \binom{c}{n} = \sum_{c=0}^{C_l} \Gamma_l^c \left[q(1-q)c + q^2 c^2 \right] \\ &= q(1-q) \langle c \rangle_l + q^2 \langle c^2 \rangle_l. \end{aligned} \quad (85)$$

Comparing with Eqs.(65), (69), and (67) we find for the cumulant tensor elements,

$$\kappa_1^N = \sum_l q \langle c_l \rangle \psi_l^{(1,1;0)} = q (\kappa_1^N)_{4\pi}, \quad \text{and} \quad (86)$$

$$\kappa_2^{B,N} = \sum_l q b_l \langle c_l \rangle \psi_l^{(1,2;0)} = q (\kappa_1^{B,N})_{4\pi}, \quad \text{etc.}, \quad (87)$$

$$\begin{aligned} \kappa_2^{N,N} &= \sum_l \left[q(1-q) \langle c_l \rangle + q^2 \langle c_l^2 \rangle \right] \psi_l^{(1,1;0)} - q^2 \langle c_l^2 \rangle \psi_l^{(2,2;0)} \\ &= q(\kappa_1^N)_{4\pi} - q^2 (\kappa_1^N)_{4\pi} + q^2 (\kappa_2^{N,N})_{4\pi}. \end{aligned} \quad (88)$$

The index ‘ 4π ’ denotes quantities that would be measured by the ideal detector with full 4π acceptance. Substituting into Eq. (76), we obtain the acceptance scaling formula from Ref. [23]:

$$\omega^{acc} = 1 - q + q\omega^{4\pi}. \quad (89)$$

VII. FINITE VOLUME CORRECTIONS

Considering finite system size effects on distributions, we leave the region where the thermodynamic limit approximation is valid. Chemical potentials μ_j and temperature T do not correspond to the physical ones, which would be found in the GCE, anymore, but have to be thought of as Lagrange multipliers, used to maximize the partition function for a given (micro)canonical state. First we derive some volume dependent corrections terms, and then find a condition that defines the correct values of μ_j and T . The correct choice allows to write down the thermodynamical potentials, the Helmholtz free energy F for CE, and the entropy S for the MCE, in terms of the generalized partition function. Some general criterion for the validity of the expansion is given. We will compare CE and MCE results with scenarios which are accessible to analytical methods in Section VII C.

A. Gram-Charlier Expansion

In Section III we have shown that in the thermodynamic limit any equilibrium multiplicity distribution can be approximated by a Gaussian Eq.(21). Further parameters, describing the shape of the distribution, skewness (κ_3), or excess (κ_4), tend to zero as volume is increased. We return to a generalized version of Eq.(19) with a number J of conserved quantities and use Gram-Charlier expansion [39]. Here we need to explicitly find the inverse square root σ^{-1} of κ_2 for the calculation of the normalized cumulants λ_n Eq.(17), see Appendix D. For every considered charge one will pick up an additional factor of \sqrt{V} from the volume element $d\theta_j$, Eq.(16),

$$\begin{aligned} \mathcal{Z}^{Q_j} \simeq \frac{Z}{V^{J/2} \det |\sigma|} \left[\prod_{j=1}^J \int_{-\infty}^{\infty} \frac{d\theta_j}{2\pi} \right] \exp \left[- i\xi^j \theta_j - \frac{\theta^j \theta_j}{2!} \right. \\ \left. + \sum_{n=3}^{\infty} i^n V^{-\frac{n}{2}+1} \frac{\lambda_n^{j_1, j_2, \dots, j_n}}{n!} \theta_{j_1} \theta_{j_2} \dots \theta_{j_n} \right]. \quad (90) \end{aligned}$$

Expanding the exponential in terms of powers in volume, we find:

$$\begin{aligned} \mathcal{Z}^{Q^j} \simeq & \frac{Z}{V^{J/2} \det |\sigma|} \left[\prod_{j=1}^J \int_{-\infty}^{\infty} \frac{d\theta_j}{2\pi} \right] \exp \left[-i\xi^j \theta_j - \frac{\theta^j \theta_j}{2!} \right] \times \left[1 + \frac{\lambda_3^{j_1, j_2, j_3}}{3!} \frac{i^3 \theta_{j_1} \theta_{j_2} \theta_{j_3}}{V^{1/2}} \right. \\ & \left. + \frac{\lambda_4^{j_1, j_2, j_3, j_4}}{4!} \frac{i^4 \theta_{j_1} \theta_{j_2} \theta_{j_3} \theta_{j_4}}{V} + \frac{1}{2!} \frac{\lambda_3^{j_1, j_2, j_3}}{3!} \frac{\lambda_3^{j_4, j_5, j_6}}{3!} \frac{i^6 \theta_{j_1} \dots \theta_{j_6}}{V} + \mathcal{O}(V^{-3/2}) \right]. \end{aligned} \quad (91)$$

Correction terms in Eq. (91) can be obtained by differentiation of $\exp[-i\xi^j \theta_j]$ with respect to ξ^j . One can thus reverse the order by first integrating and then again differentiating. Using generalized Hermite polynomials,

$$(H_n(\xi))_{j_1, j_2, \dots, j_n} = (-1)^n \exp \left[\frac{\xi^j \xi_j}{2} \right] \frac{d^n}{d\xi_{j_1} d\xi_{j_2} \dots d\xi_{j_n}} \exp \left[-\frac{\xi^j \xi_j}{2} \right], \quad (92)$$

with the adjusted shorthand notation,

$$h_3(\xi) = \frac{\lambda_3^{j_1, j_2, j_3}}{3!} (H_3(\xi))_{j_1, j_2, j_3}, \quad (93)$$

$$h_4(\xi) = \frac{\lambda_4^{j_1, j_2, j_3, j_4}}{4!} (H_4(\xi))_{j_1, j_2, j_3, j_4} + \frac{1}{2!} \frac{\lambda_3^{j_1, j_2, j_3}}{3!} \frac{\lambda_3^{j_4, j_5, j_6}}{3!} (H_6(\xi))_{j_1, \dots, j_6}, \quad (94)$$

$$\begin{aligned} h_5(\xi) = & \frac{\lambda_5^{j_1, \dots, j_5}}{5!} (H_5(\xi))_{j_1, \dots, j_5} + \frac{\lambda_3^{j_1, j_2, j_3}}{3!} \frac{\lambda_4^{j_1, j_2, j_3, j_4}}{4!} (H_7(\xi))_{j_1, \dots, j_7} \\ & + \frac{1}{3!} \frac{\lambda_3^{j_1, j_2, j_3}}{3!} \frac{\lambda_3^{j_4, j_5, j_6}}{3!} \frac{\lambda_3^{j_7, j_8, j_9}}{3!} (H_9(\xi))_{j_1, \dots, j_9}, \end{aligned} \quad (95)$$

the partition function for finite volume can be approximated by:

$$\mathcal{Z}^{Q^j} \simeq Z \frac{e^{-\frac{\xi^j \xi_j}{2}}}{(2\pi V)^{J/2} \det |\sigma|} \left[1 + \frac{h_3(\xi)}{\sqrt{V}} + \frac{h_4(\xi)}{V} + \frac{h_5(\xi)}{V^{3/2}} + \mathcal{O}(V^{-2}) \right]. \quad (96)$$

Considering the simplest case of only one conserved charge, it is evident from Eq.(92), that the first order correction term in Eq.(96) is a polynomial of order 3 in ξ , while the second order correction term is a polynomial of order 4, etc. Hence for large values of ξ , e.g. a multiplicity state far from the peak of the distribution will lead to a bad approximation, and even to negative values for $P(N)$. The validity of this approximation is thus restricted to the central region of the distribution. We will compare CE and MCE results with scenarios which are accessible to analytical methods in Section VII C. In order to distinguish approximations which include corrections up to different orders in volume in Eq.(96), we denote the asymptotic solution as CLT (central limit theorem), including terms up to $\mathcal{O}(V^{-1/2})$ as GC3 (Gram-Charlier 3), including terms up to $\mathcal{O}(V^{-1})$ as GC4, and including terms up to $\mathcal{O}(V^{-3/2})$ as GC5.

B. Chemical and Thermal Equilibrium

Here we address the question of how to choose the optimal values for T and μ_j . If exact solutions were available, distributions, however *not* thermodynamic potentials, would be independent of this choice (see as well Section VIII). Our postulate is that our (micro)canonical equilibrium state should be as well the most likely state in the GCE. For an isotropic momentum distribution, the macroscopic state $\vec{P} = \vec{0}$ is always the most probable state, since all odd cumulants involving only momenta vanish $\kappa_1^{p_x} = \kappa_3^{p_x, p_x, p_x} = \dots = 0$. On the other hand, we know that the expansion works best around the peak of the distribution. So we choose T and μ_j such that we maximize the partition function at some point equilibrium Q_j^{eq} . Taking terms up to $\mathcal{O}(V^{-1/2})$ into account, the first derivative of the partition function Eq.(12) reads:

$$\frac{\partial \mathcal{Z}^{Q_j}}{\partial Q_j} = \frac{e^{-\frac{\xi_j^j \xi_j}{2}}}{(2\pi)^{J/2} V^{(J+1)/2} \det |\sigma|} \left[\xi_k (\sigma^{-1})^k_j + \frac{\lambda_3^{k_1, k_2, k_3}}{3! \sqrt{V}} (\sigma^{-1})^{k_4}_j H_4(\xi)_{k_1, k_2, k_3, k_4} + \mathcal{O}(V^{-1}) \right]. \quad (97)$$

The chemical potentials should be chosen such that the first derivative Eq.(97) of \mathcal{Z}^{Q_j} with respect to the conserved quantities Q_j vanishes, hence we maximize Eq.(96) at the point Q_j^{eq} :

$$\left. \frac{\partial \mathcal{Z}^{Q_j}}{\partial Q_j} \right|_{Q_j^{eq}} = \vec{0}. \quad (98)$$

Using only the asymptotic solution, valid in the thermodynamic limit, this condition leads to:

$$\xi_k = (Q_j - V \kappa_{1,j}) (\sigma^{-1})^j_k = \vec{0}. \quad (99)$$

Hence the partition function is maximal at the point $Q_j^{eq} = V \kappa_{1,j}$. Charge and energy density correspond thus to the GCE values, and $\mu^j \rightarrow \mu_{gce}^j$ and $T \rightarrow T_{gce}$. While when taking the first finite volume correction term in Eq.(97) into account we obtain:

$$\xi_k (\sigma^{-1})^k_j + \frac{\lambda_3^{k_1, k_2, k_3}}{3! \sqrt{V}} (\sigma^{-1})^{k_4}_j H_4(\xi)_{k_1, k_2, k_3, k_4} = \vec{0}, \quad (100)$$

rather than Eq.(99), and $\mu^j \neq \mu_{gce}^j$, and $T \neq T_{gce}$. The recipe for calculation of distributions for finite volume system thus goes as follows. One should find chemical potentials that satisfy condition (98). Then keep in mind that they are chemical potentials only in the thermodynamic limit, while for finite volume they are simply Lagrange multipliers. Then one should calculate the normalization $\mathcal{Z}^{Q_j, eq}$ and the distribution $\tilde{\mathcal{Z}}^{Q_j, eq}$ using chemical potentials and temperature

obtained from Eq.(98). Their ratio gives the distribution $P(N|Q^{j,eq})$ of particles of the selected species. A technical comment is in order. From Eq.(97) it is evident that the first order correction term to the derivative of the partition function is a polynomial of order 4 in ξ , while the second one is of order 5, etc. It is therefore crucial to find in numerical calculations the correct maximum.

C. Quality of Approximation

To test the quality of our approximation for multiplicity distributions at finite volume, Eq.(96), for (very) small systems, we compare to analytical solutions for a CE classical particle-anti-particle gas, and a classical MCE (without momentum conservation) ultra-relativistic gas. The exact solutions are given by Eq.(121) and Eq.(136) respectively. Figure 1 shows on the top row the multiplicity distribution of positively charged particles for various system sizes in the exact form Eq.(121) and in different orders of approximation Eq.(96). On the bottom row the ratio of approximation to exact solution is taken. In figure 2 the same physical system is shown for a (relatively large) positive net-charge. Due to a one-to-one correspondence between the distributions of negatively (suppressed) and positively (enhanced) particles we find the distribution $P(N_+)$ generally more narrow than in the case of a neutral system. In particular towards the edge of the body of the distribution the approximation is worse. For the MCE massless gas we compare again approximations to $P(N)$ and ratios to the exact solution on top and bottom row of figures 3 respectively for different system sizes. A few general comments attempt to summarize these figures. Our first observation is that indeed as system size is increased we find a better description of the central region in terms of the asymptotic solution CLT (Gaussian with width given by Eq.(28)). Our second observation is that even for systems with a very small number (in the order of 5) of produced particles we find a good approximation in terms of Gram-Charlier expansion. In particular GC5 provides a very accurate description of the central region with deviations in the order of a few percent. This is quite remarkable given the fact that multiplicity distributions for such small systems are not smooth and continuous functions of multiplicity, while our approximation Eq.(96) is. Furthermore we have implicitly introduced the concepts of chemical potential and temperature for systems with small particle number, which may be in contradiction to the common belief that these parameters can only be meaningful when the number of involved particles becomes very large, i.e. in the

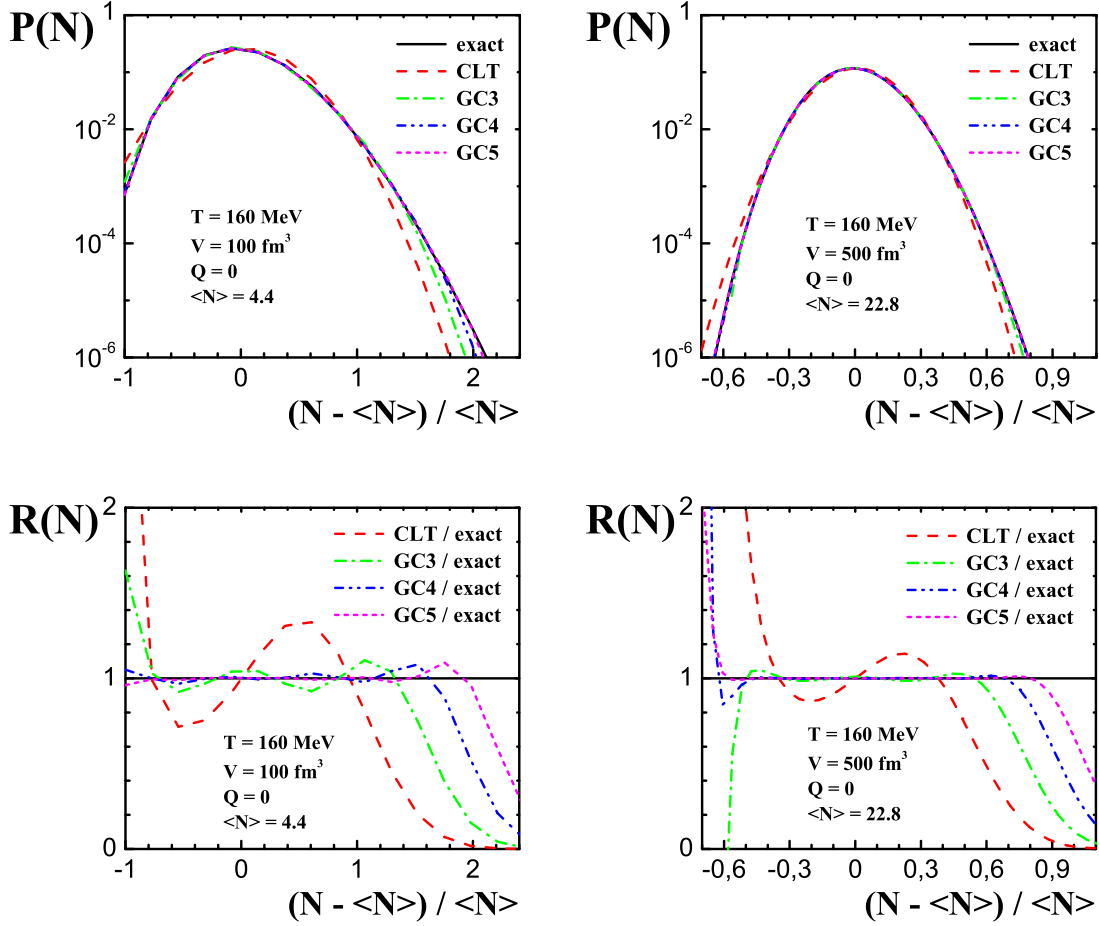


FIG. 1: *top*: Boltzmann CE π^+ multiplicity distribution for $T = 160 \text{ MeV}$, $Q = 0$ for $V = 100 \text{ fm}^3$ (*left*), and $V = 500 \text{ fm}^3$ (*right*). Exact solution (solid), and in CLT (dash), GC3 (dash-dot), GC4 (dash-dot-dot), and GC5 (dot) approximations. *bottom*: same as top, but ratio of exact solution to approximation.

thermodynamic limit. Our last observation is that indeed (see bottom rows of figures 1-3) finite volume corrections given in terms of polynomials lead only to good results for the central region of the distribution.

To give an estimate for a region in which the approximation is reliable, we note that our finite volume approximation scheme begins to break down when the first expansion term in Eq. (96) becomes unity. In the one-dimensional case this would be:

$$\frac{h_3(\xi_{\max})}{\sqrt{V}} \sim \mathcal{O}(1) . \quad (101)$$

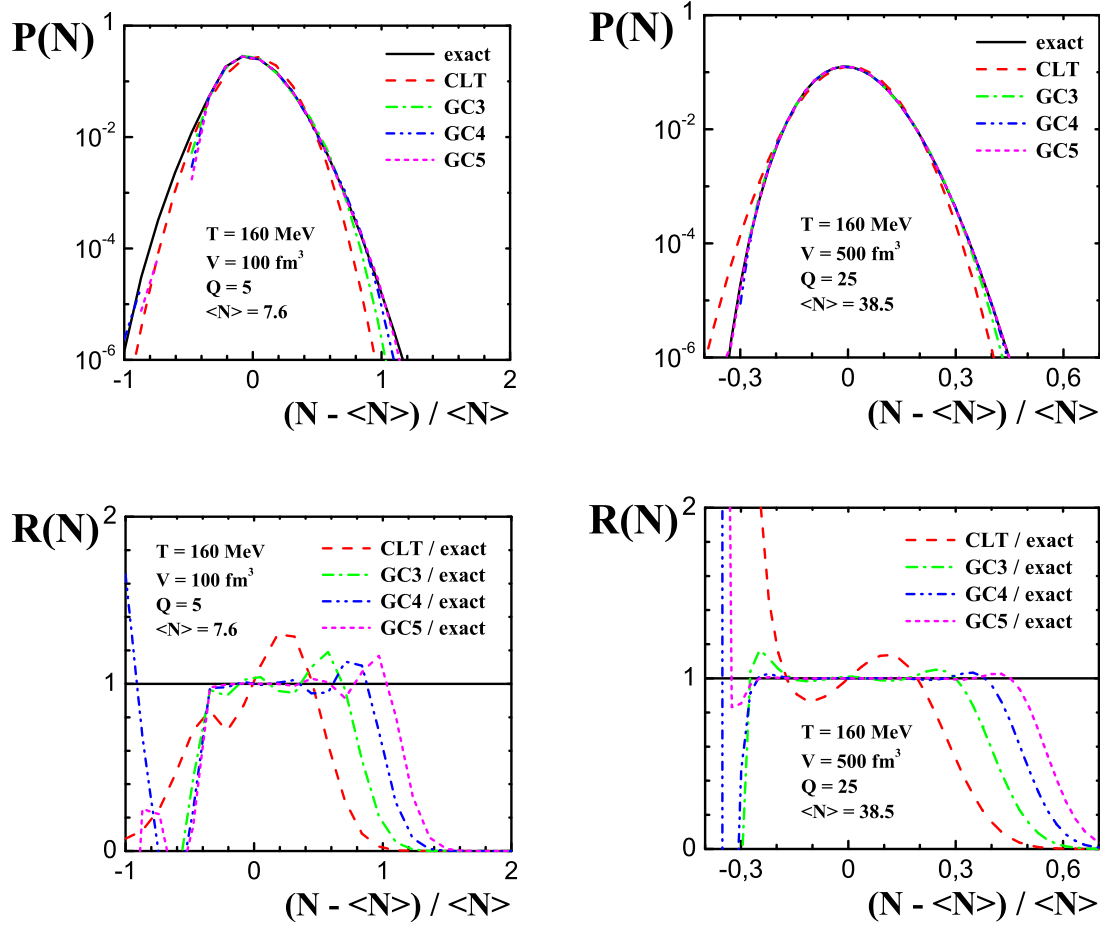


FIG. 2: same as 1, but with $Q = 5$ (left), and $Q = 25$ (right)

Approximating the Hermite polynomial $H_3 \sim \xi^3$, one can get an estimate for ξ_{max} .

$$\xi_{max} \simeq \left(\frac{3!}{\lambda_3} \right)^{1/3} V^{1/6}. \quad (102)$$

While, when switching back to the definition of $\xi = \frac{Q - V\kappa_1}{\sigma\sqrt{V}}$, Eq.(18), the width of the central region can be estimated by:

$$\frac{|Q - V\kappa_1|_{max}}{\sigma} \simeq \left(\frac{3!}{\lambda_3} \right)^{1/3} V^{2/3}. \quad (103)$$

Hence the width of the central region scales as $V^{2/3}$, while the width of the curve should scale as $V^{1/2}$ and our approximation should be quite good. Even though larger volumes work better, they will still be sufficiently small enough to allow for calculation of distributions relevant for heavy ion collisions. However we want to stress that there is no simple criterion for what

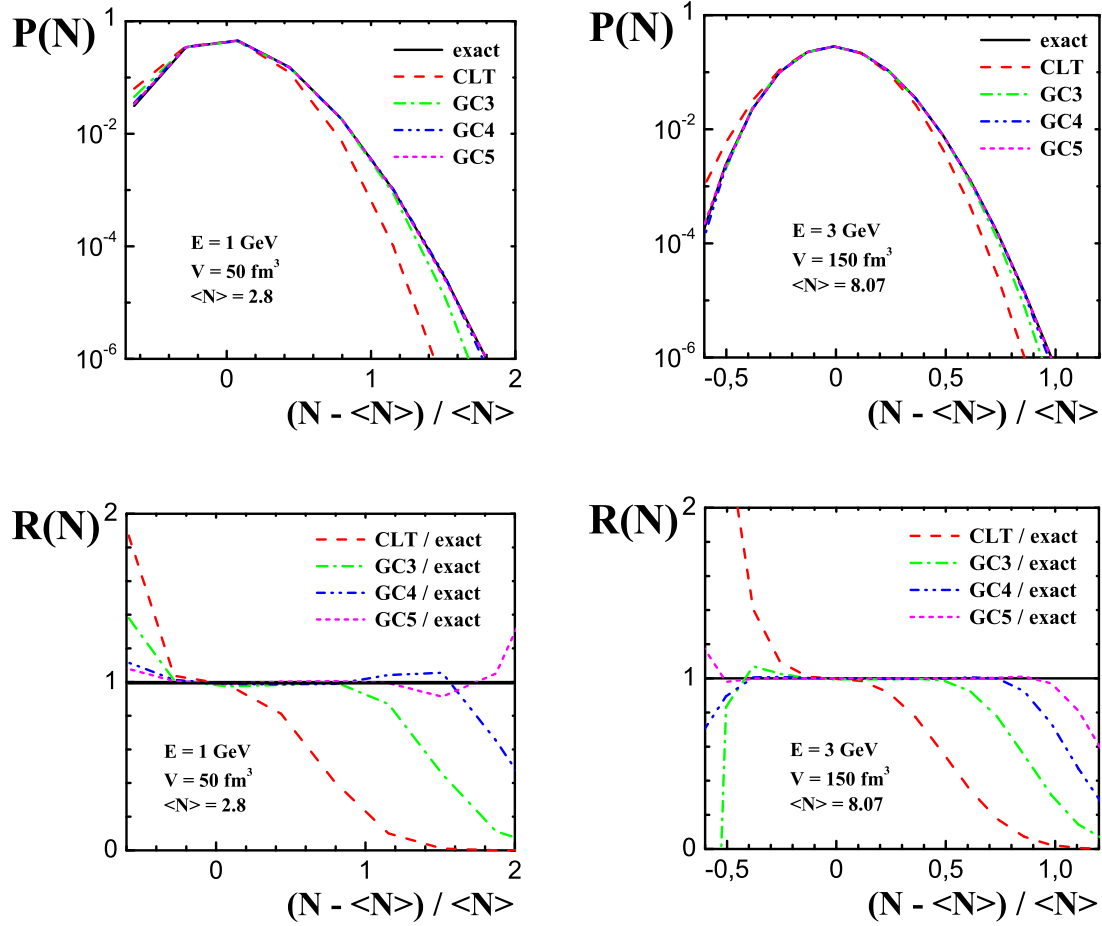


FIG. 3: *top*: Particle number distribution in the MCE without momentum conservation for a ultra-relativistic Boltzmann gas. $E = 1\text{ GeV}$, $V = 50\text{ fm}^3$ (*left*), and $E = 3\text{ GeV}$, $V = 150\text{ fm}^3$ (*right*). Exact solution (dash), and in CLT (dash-dot), GC3 (dash-dot), GC4 (dash-dot-dot), and GC5 (dot) approximations. *bottom*: same as top, but ratio of exact solution to approximation.

is a ‘small’ or a ‘large’ volume for a particular physical system. Formally the existence and finiteness of the (at least) first three cumulants κ is sufficient for application of the asymptotic solution [39]. Considering the simple case of a multiplicity distribution of BE particles in the GCE we find from Eq.(8) $\kappa_1^N < \kappa_2^{N,N} < \kappa_3^{N,N,N} < \dots$. Hence we find, in particular when finite chemical potentials are involved, cumulants growing with order, implying that apart from mean and variance further parameters like skewness and excess [38] of the distribution will remain important quantities.

D. Temperature and Chemical Potentials in MCE and CE

The introduction of chemical potentials in the CE and temperature in the MCE was first and foremost a mathematical trick which allowed to conveniently integrate partition functions for which otherwise no analytical solution could be obtained. However our generalized partition function is self-consistent and not in contradiction to the common definition of temperature and chemical potential. We will show in the following that our definition of T and μ through Eq.(98) coincides with expressions well known from textbooks [33].

1. Canonical Ensemble

The canonical partition function known from textbooks and our generalized version are connected as follows (see Appendix B):

$$Z^{CE}(Q, V, T) \equiv Z^Q(Q, V, T, \mu_Q) e^{-Q \frac{\mu_Q}{T}}. \quad (104)$$

The Helmholtz free energy F is the thermodynamic potential relevant for CE,

$$F \equiv -T \ln Z^{CE}. \quad (105)$$

Using the first law of thermodynamics $dE = TdS - PdV + \mu_Q dQ$, and $F = E - TS$, where P is the pressure, and E and S are total energy and entropy, respectively, we can write for the differential $dF = -SdT - PdV + \mu_Q dQ$. The effective chemical potential μ_Q associated with conserved (electric) charge Q is thus defined by:

$$\left(\frac{\partial F}{\partial Q} \right)_{V,T} = -T \frac{\frac{\partial Z^Q}{\partial Q} e^{-Q \frac{\mu_Q}{T}} - \frac{\mu_Q}{T} Z^Q e^{-Q \frac{\mu_Q}{T}}}{Z^Q e^{-Q \frac{\mu_Q}{T}}} = \mu_Q, \quad (106)$$

where we have used condition (98). Thus the correct choice for the effective chemical potential is $\frac{\partial Z^Q}{\partial Q} = 0$, which coincides with $\mu_Q = \left(\frac{\partial F}{\partial Q} \right)_{V,T}$. In the thermodynamic limit this is equivalent to, Eq.(99), $\mu_Q \rightarrow \mu_{Q,ge}$. The subscript in Eq.(106) is used to indicate that the derivative with respect to the conserved charge has to be taken at fixed values of V and T . In the non-relativistic case, where particle number N , rather than quantum numbers, is conserved, the corresponding relation to Eq.(106) would be $\left(\frac{\partial F}{\partial N} \right)_{V,T} = \mu_N$ [33]. For a detailed presentation of CE calculations please see Section VIII.

2. Microcanonical Ensemble

The common MCE partition function can be obtained by integration of the GGPF $\mathcal{Z}(\phi_E)$ and multiplication by the inverse Boltzmann factor (see Appendix C for details),

$$Z^{MCE}(E, V) \equiv \mathcal{Z}^E(E, V, T) e^{\frac{E}{T}}. \quad (107)$$

The relevant thermodynamic potential in the MCE is the entropy S ,

$$S \equiv \ln Z^{MCE}. \quad (108)$$

The effective temperature is [33]:

$$\left(\frac{\partial S}{\partial E} \right)_V = \frac{\frac{\partial \mathcal{Z}^E}{\partial E} e^{\frac{E}{T}} + \frac{1}{T} \mathcal{Z}^E e^{\frac{E}{T}}}{\mathcal{Z}^E e^{\frac{E}{T}}} = \frac{1}{T}, \quad (109)$$

where we used condition (98), $\frac{\partial \mathcal{Z}^E}{\partial E} = 0$. The subscript in Eq.(109) is used to indicate that the derivative with respect to E has to be taken at fixed volume V . Thus Eq.(109) resembles the optimal choice of an effective temperature for our approximation scheme. In the thermodynamic limit, $V \rightarrow \infty$, we find $T \rightarrow T_{gce}$, due to Eq.(99).

3. Grand Canonical Ensemble

Conventionally, e.g. in textbooks, first the MCE is introduced. Summation over energy, with temperature being a Lagrange multiplier, used to maximize the entropy, introduces the CE. Additionally dropping the constraint of exact charge conservation leads to the GCE. Here the chemical potential μ_Q is the Lagrange multiplier. For the MCE (without momentum conservation) and one conserved charge Q the GCE and CE partition functions are defined by:

$$Z^{GCE}(V, T, \mu_Q) = \sum_Q e^{Q \frac{\mu_Q}{T}} Z^{CE}(V, T, Q) = \sum_{Q, E} e^{Q \frac{\mu_Q}{T}} e^{-\frac{E}{T}} Z^{MCE}(V, E, Q). \quad (110)$$

While in our notation this line would read:

$$\mathcal{Z}(V, T, \mu_Q) = \sum_Q \mathcal{Z}^Q(V, T, \mu_Q) = \sum_{Q, E} \mathcal{Z}^{E, Q}(V, T, \mu_Q). \quad (111)$$

The thermodynamic potential for the GCE is the grand potential Ω :

$$\Omega \equiv -T \ln Z^{GCE} = -T \ln \mathcal{Z} |_{\phi=0}. \quad (112)$$

Again, if exact solutions of the canonical or microcanonical partition functions were available this reversal would not have been necessary. However this redefinition of the GCE partition function is entirely consistent and simplifies calculations considerably. Whenever an exact solution to our generalized partition function is possible, all the above relations would hold exactly. We believe this interpretation of the GCE partition function as the generating (or characteristic) function of a statistical system to be quite useful, even in more general cases than the one presented here.

VIII. THE SIMPLEST EXAMPLE

A. An Exact Solution

To clarify the procedure described in the previous sections, let us consider a simple example of an ideal Boltzmann particle-anti-particle gas ($q_+ = 1$, $q_- = -1$). In this simple case the calculations discussed above can be done explicitly. The GCE partition function reads:

$$Z = \sum_{n_+, n_- = 0}^{\infty} \frac{z_+^{n_+} z_-^{n_-}}{n_+! n_-!} \equiv \sum_{n_+, n_- = 0}^{\infty} Z(n_+, n_-) = \exp[z_+ + z_-] \equiv \exp\left[2z \cosh\left(\frac{\mu_Q}{T}\right)\right], \quad (113)$$

where $z_{\pm} = z \exp(\pm \mu_Q/T)$, and z is a single particle partition function in MB approximation:

$$z = \frac{gV}{2\pi^2} \int_0^{\infty} p^2 dp \exp\left(-\frac{\sqrt{p^2 + m^2}}{T}\right) = \frac{gV}{2\pi^2} m^2 T K_2\left(\frac{m}{T}\right), \quad (114)$$

where g and m are respectively the degeneracy factor and particle mass, and K_2 is the modified Hankel function. The conserved charge Q is just the difference of N_+ and N_- . To be definite we discuss the distribution $P(N_+)$ of positively charged particles. In the GCE it can be easily found:

$$\begin{aligned} P(N_+) &= \frac{\text{all states with } N_+ \text{ particles}}{\text{all states}} = \frac{\sum_{n_+, n_- = 0}^{\infty} Z(n_+, n_-) \delta(N_+ - n_+)}{\sum_{n_+, n_- = 0}^{\infty} Z(n_+, n_-)} \\ &= \frac{z_+^{N_+}}{N_+!} \exp(-z_+). \end{aligned} \quad (115)$$

As can be seen it has the form of the Poisson distribution with the following first two moments:

$$\langle N_+ \rangle_{g.c.e.} = \sum_{N_+} N_+ P(N_+) = z_+, \quad \langle N_+^2 \rangle_{g.c.e.} = \sum_{N_+} N_+^2 P(N_+) = z_+^2 + z_+. \quad (116)$$

This gives the GCE scaled variance:

$$\omega_{g.c.e.}^+ \equiv \frac{\langle N_+^2 \rangle_{g.c.e.} - \langle N_+ \rangle_{g.c.e.}^2}{\langle N_+ \rangle_{g.c.e.}} = 1. \quad (117)$$

In the thermodynamic limit $z \rightarrow \infty$ the Poisson distribution (115) can be transformed into the Gauss one. Using Stirling's formula, $N_+! \simeq \sqrt{2\pi N_+} \exp(N_+ \ln N_+ - N_+)$, for $N_+ \gg 1$ one finds at $|N_+ - z| \ll z$:

$$P(N_+) = \frac{z_+^{N_+}}{N_+!} \exp(-z_+) \simeq \frac{1}{\sqrt{2\pi} z_+} \exp\left[-\frac{(N_+ - z_+)^2}{2z_+}\right]. \quad (118)$$

The distribution of net charge $P(Q)$ and the joint distribution $P(N_+, Q)$ can be found by the straightforward calculations:

$$P(Q) = \frac{1}{Z} \sum_{n_+, n_- = 0}^{\infty} \delta(Q - [n_+ - n_-]) \frac{z_+^{n_+} z_-^{n_-}}{n_+! n_-!} = \frac{e^{Q\mu_Q/T}}{Z} I_Q(2z), \quad (119)$$

$$P(N_+, Q) = \frac{1}{Z} \sum_{n_+, n_- = 0}^{\infty} \delta(N_+ - n_+) \delta(Q - [n_+ - n_-]) \frac{z_+^{n_+} z_-^{n_-}}{n_+! n_-!} = \frac{e^{Q\mu_Q/T}}{Z} \frac{z^{2N_+ - Q}}{N_+!(N_+ - Q)!}, \quad (120)$$

where $I_Q(2z)$ is the modified Bessel function. The Eqs. (119, 120) are the simplest appearance of the general Eqs. (2, 4). One can also notice that for this example the CE partition function from the Eq. (2) equals the modified Bessel function $Z^Q = I_Q(2z)$.

From the Eqs. (119, 120) one finds the CE particle number distribution (see the Eq. (5)):

$$P(N_+|Q) = \frac{P(N_+, Q)}{P(Q)} = \frac{z^{N_+}}{N_+!} \frac{z^{N_+ - Q}}{(N_+ - Q)!} [I_Q(2z)]^{-1}. \quad (121)$$

The first and second moments of the CE multiplicity distribution can be easily found [28]:

$$\langle N_+ \rangle_{c.e.} = \sum_{N_+ = Q}^{\infty} N_+ P(N_+|Q) = z \frac{I_{Q-1}(2z)}{I_Q(2z)}, \quad (122)$$

$$\langle N_+^2 \rangle_{c.e.} = \sum_{N_+ = Q}^{\infty} N_+^2 P(N_+|Q) = z \frac{I_{Q-1}(2z)}{I_Q(2z)} + z^2 \frac{I_{Q-2}(2z)}{I_Q(2z)}. \quad (123)$$

This leads to the CE scaled variance:

$$\omega_{c.e.}^+ \equiv \frac{\langle N_+^2 \rangle_{c.e.} - \langle N_+ \rangle_{c.e.}^2}{\langle N_+ \rangle_{c.e.}} = 1 - z \left[\frac{I_{Q-1}(2z)}{I_Q(2z)} - \frac{I_{Q-2}(2z)}{I_{Q-1}(2z)} \right]. \quad (124)$$

In the thermodynamic limit when $z \rightarrow \infty$ and $Q/2z = y$ one finds [28]:

$$\langle N_+ \rangle_{c.e.} = z (y + \sqrt{1 + y^2}), \quad \omega_{c.e.}^+ = \frac{1}{2} - \frac{y}{2\sqrt{1 + y^2}}. \quad (125)$$

Let us compare the results for the average particle number $\langle N_+ \rangle$ and for the scaled variance ω^+ calculated in the GCE and CE. We fix the V and T parameters to be the same in the GCE and CE. The relation between the chemical potential μ_Q of the GCE and the charge Q of the CE formulation is obtained from the following requirement:

$$Q = \langle Q \rangle = \langle N_+ \rangle_{g.c.e.} - \langle N_- \rangle_{g.c.e.} = z_+ - z_- = 2z \sinh\left(\frac{\mu_Q}{T}\right). \quad (126)$$

This gives, $\exp(\mu_Q/T) = y + \sqrt{1+y^2}$, and leads to $\langle N_+ \rangle_{c.e.} \simeq \langle N_+ \rangle_{g.c.e.}$, which means the thermodynamic equivalence of the CE and GCE. Comparing Eqs. (117) and (125) one finds $\omega_{c.e.}^+ \neq \omega_{g.c.e.}^+$, thus, the scaled variances are not equivalent even in the thermodynamic limit.

We have used the GCE distributions $P(Q)$ and $P(N_+, Q)$ to calculate the CE distribution $P(N_+|Q)$. The Eq. (121) demonstrates that the μ_Q -dependence has completely disappeared in $P(N_+|Q)$ if exact analytical results in the GCE for $P(Q)$ (119) and $P(N_+, Q)$ (120) are used. This means that the GCE with arbitrary value of μ_Q (and, thus, the corresponding value of $\langle Q \rangle$) can be used for the *exact* CE calculations of $P(N_+|Q)$.

B. The Saddle Point Expansion

We will discuss now the *approximate* calculations of $P(Q)$ (119) and $P(N_+, Q)$ (120) in the thermodynamic limit. We replace the corresponding delta-functions which fix the value of net charge, Q , and positively charged particle number, N_+ by their Fourier representations as the ϕ_Q and ϕ_+ integrations. This method can be used for more complicated cases when exact analytical results can not be obtained. In these asymptotic calculations the role of the chemical potential will be demonstrated. The Eq. (119) can be rewritten as follows:

$$\begin{aligned} P(Q) &= \frac{1}{Z} \int_{-\pi}^{\pi} \frac{d\phi_Q}{2\pi} \exp(-iQ\phi_Q) \mathcal{Z}(\phi_Q) \\ &= \frac{1}{Z} \int_{-\pi}^{\pi} \frac{d\phi_Q}{2\pi} \exp(-iQ\phi_Q) \exp \left[z \exp\left(\frac{\mu_Q}{T} + i\phi_Q\right) + z \exp\left(-\frac{\mu_Q}{T} - i\phi_Q\right) \right]. \end{aligned} \quad (127)$$

In thermodynamic limit, $z \rightarrow \infty$, one can expand $\exp(\pm[\mu_Q/T + i\phi_Q])$ in the Taylor series and leave only the terms up to ϕ_Q^2 , because for $z \rightarrow \infty$ the main contribution comes from the

$\phi_Q = 0$ region. Then the distribution $P(Q)$ becomes a Gaussian:

$$\begin{aligned} P(Q) &= \int_{-\infty}^{\infty} \frac{d\phi_Q}{2\pi} \exp \left[-i(Q - 2z \sinh(\mu_Q/T)) \phi_Q - z \cosh(\mu_Q/T) \phi_Q^2 + \dots \right] \\ &\simeq P_G(Q) = [4\pi z \cosh(\mu_Q/T)]^{-1/2} \exp \left[-\frac{(Q - 2z \sinh(\mu_Q/T))^2}{4z \cosh(\mu_Q/T)} \right]. \end{aligned} \quad (128)$$

For $P(N_+, Q)$ at $z \rightarrow \infty$ one similarly finds:

$$\begin{aligned} P(N_+, Q) &= \frac{1}{Z} \int_{-\pi}^{\pi} \frac{d\phi_Q}{2\pi} e^{-iQ\phi_Q} \int_{-\pi}^{\pi} \frac{d\phi_+}{2\pi} e^{-iN_+\phi_+} \mathcal{Z}(\phi_Q, \phi_+) \\ &= \frac{1}{Z} \int_{-\pi}^{\pi} \frac{d\phi_Q}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi_+}{2\pi} \exp \left[-iQ\phi_Q - iN_+\phi_+ + z \exp \left(\frac{\mu_Q}{T} + i\phi_Q + i\phi_+ \right) + z \exp \left(-\frac{\mu_Q}{T} - i\phi_Q \right) \right] \\ &\simeq P_G(N_+, Q) \equiv \frac{1}{Z} \int_{-\infty}^{\infty} \frac{d\phi_Q}{2\pi} \int_{-\infty}^{\infty} \frac{d\phi_+}{2\pi} \exp \left[i\phi_Q (z_+ - z_- - Q) + i\phi_+ (z_+ - N_+) \right. \\ &\quad \left. - (z_+ + z_-) \frac{\phi_Q^2}{2} - z_+ \frac{\phi_+^2}{2} - z_+ \phi_Q \phi_+ \right]. \end{aligned} \quad (129)$$

The integration over ϕ_Q and ϕ_+ in Eq. (129) gives:

$$\begin{aligned} P_G(N_+, Q) &= \frac{1}{2\pi z} \exp \left[-\frac{1}{2z} \exp(\mu_Q/T) (Q - \langle Q \rangle)^2 + \frac{1}{z} \exp(\mu_Q/T) (Q - \langle Q \rangle) (N_+ - \langle N_+ \rangle) \right. \\ &\quad \left. - \frac{1}{z} \cosh(\mu_Q/T) (N_+ - \langle N_+ \rangle)^2 \right], \end{aligned} \quad (130)$$

where $\langle Q \rangle$ and $\langle N_+ \rangle$ in Eq. (130) correspond to the GCE values,

$$\langle Q \rangle = 2z \sinh(\mu_Q/T), \quad \langle N_+ \rangle = z \exp(\mu_Q/T). \quad (131)$$

The Eq. (130) has the form of Bivariate Normal Distribution i.e. Gauss (Normal) distribution in two dimensions [38]. The CE distribution (121) is then approximated as,

$$\begin{aligned} P(N_+|Q) &= \frac{P(N_+, Q)}{P(Q)} \simeq P_G(N_+|Q) \equiv \frac{P_G(N_+, Q)}{P_G(Q)} \\ &= \frac{1}{\sqrt{\pi z \cosh^{-1}[\mu_Q/T]}} \exp \left[-\frac{\cosh(\mu_Q/T)}{z} (N_+ - \langle N_+ \rangle)^2 \right. \\ &\quad \left. + \frac{\exp(\mu_Q/T)}{z} (N_+ - \langle N_+ \rangle) (Q - \langle Q \rangle) - \frac{\exp(2\mu_Q/T)}{2z \cosh(\mu_Q/T)} (Q - \langle Q \rangle)^2 \right]. \end{aligned} \quad (132)$$

Comparing the distributions (128) and (130) with the exact expressions for $P(Q)$ (119), and $P(N_+, Q)$ (120) one can notice that the Eqs. (119), and (120) contain the dependance on the chemical potential μ_Q just as the factor $e^{\mu_Q/T}$. Thus if one succeeds in the exact calculations the resulting CE distribution $P(N_+|Q)$ (121) does not include μ_Q dependence in contrast to (132). It means that the choice of the chemical potential is irrelevant for the exact CE calculations (usually it is chosen equal to zero), while the value of μ_Q is crucial for the approximate calculations. The saddle point expansion works if μ_Q chosen to fix $\langle Q \rangle = Q$ in thermodynamic limit, i.e. for $\mu_Q/T = \text{arcsinh}(Q/2z)$. Thus the GCE should be thermodynamically equivalent to the CE with fixed Q net charge.

Let us illustrate these statements. We plotted the exact (119) and approximate (Gauss) distributions $P(Q)$ (128) for the arbitrarily chosen values $z = 20$ and $Q = 50$ with zero chemical potential $\mu_Q = 0$ (Fig. 4, left) and for the chemical potential $\mu_Q \neq 0$ that corresponds to the condition $\langle Q \rangle = Q$ (Fig. 5, left).

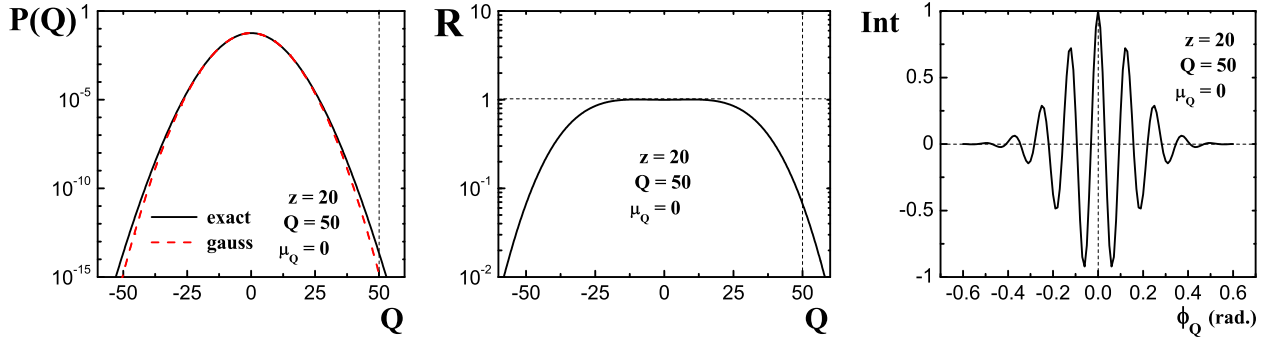


FIG. 4: The exact and approximate distribution $P(Q)$ (left), their ratio (center), and the integrant (Int) from the Eq. (128) (right) for the 'wrong' chemical potential $\mu_Q/T \neq \text{arcsinh}(Q/2z)$.

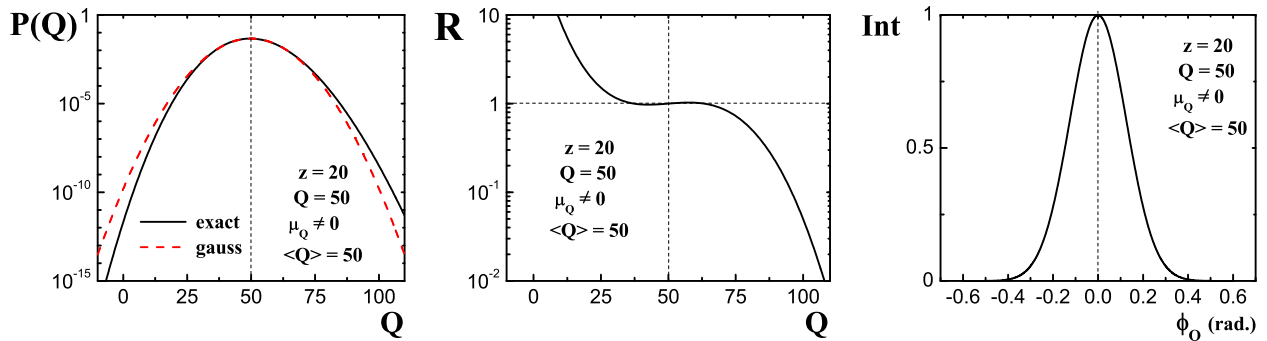


FIG. 5: The same as in Fig. 4 but for correct $\mu_Q = \text{arcsinh}(Q/2z)$.

One can see that if one chooses the wrong chemical potential $\mu_Q = 0$ than the approximate

distribution (128) is approximately 30 times smaller than the exact one (119) at the point $Q = 50$ (see the ratios $P_G(Q)/P(Q)$ Fig. 4, 5, center). For non zero chemical potential $\mu_Q \neq 0$ that satisfy the condition $\langle Q \rangle = Q$, i.e. for the chemical potential $\mu_Q/T = \text{arcsinh}(Q/2z)$, the exact and the approximate distribution almost coincide in a wide range near $Q = 50$ (see Fig. 5, left and center).

The mathematical meaning of this physical requirement, $\langle Q \rangle = Q$, is the following. The approximate formulas always have an oscillating part $\int_{-\infty}^{\infty} \exp[iA\phi - B\phi^2] d\phi = \int_{-\infty}^{\infty} \cos[A\phi] \exp[-B\phi^2] d\phi$ (the imaginary part includes $\sin[A\phi]$ that gives zero after integration because of symmetric integration bounds). This oscillating part is nonzero $A \neq 0$ for $\langle Q \rangle \neq Q$ thus we need to take the terms higher than ϕ^2 in order to make a good approximation. The integrant (Int) from the Eq. (128), $Int = \exp[-i(Q - 2z \sinh(\mu_Q/T))\phi_Q - z \cosh(\mu_Q/T)\phi_Q^2 + \dots]$, is shown in the Figs. 4, 5 (right). One can see that for $\langle Q \rangle \neq Q$ (Figs. 4, 5, center) the Int oscillates around zero fast, thus even small mistake in numerical integration, that is unavoidable in real calculations, gives the wrong result for $P(Q)$. However if we take the correct μ_Q that satisfy the condition $\langle Q \rangle = Q$, then $Int = \exp[-z \cosh(\mu_Q/T)\phi_Q^2 + \dots]$ becomes smooth and the term ϕ_Q^2 is enough (see Fig. 5, right).

At the point $\langle Q \rangle = Q$ from the Eq. (130) one finds:

$$\begin{aligned} P_G(N_+|Q) &= \frac{1}{\sqrt{\pi z \cosh^{-1}(\mu_Q/T)}} \exp \left[-\frac{(N_+ - \langle N_+ \rangle)^2}{z \cosh^{-1}(\mu_Q/T)} \right] \\ &= \frac{1}{\sqrt{2\pi \omega_{c.e.}^+ \cdot \langle N_+ \rangle}} \exp \left[-\frac{(N_+ - \langle N_+ \rangle)^2}{2 \omega_{c.e.}^+ \cdot \langle N_+ \rangle} \right]. \end{aligned} \quad (133)$$

In Eq. (133) the asymptotic values for multiplicity (valid in the GCE and CE), $\langle N_+ \rangle = z \exp[\mu_Q/T]$, and for the scaled variance in the CE, $\omega_{c.e.}^+ = [2 \exp(\mu_Q/T) \cosh(\mu_Q/T)]^{-1}$ (see Eq. (125)), have been used.

The above procedure can be generalized for the case of n integrations using the following formula:

$$\left[\prod_{i=1}^n \int_{-\infty}^{\infty} \frac{d\phi_i}{2\pi} \right] \exp \left[i \vec{A}^T \vec{\phi} - \frac{1}{2} \vec{\phi}^T B \vec{\phi} \right] = \frac{\exp \left[-\frac{1}{2} \vec{A}^T B^{-1} \vec{A} \right]}{(2\pi)^{n/2} \sqrt{\det |B|}}, \quad (134)$$

where T means transposed vector and B^{-1} is the inverse of a nonsingular matrix B . For

example, the distribution $P(N_+, Q)$ has the following parameters:

$$\vec{A} = \begin{pmatrix} z_+ - z_- - Q \\ z_+ - N_+ \end{pmatrix}, \quad \vec{\phi} = \begin{pmatrix} \phi_Q \\ \phi_+ \end{pmatrix}, \quad B = \begin{pmatrix} z_+ + z_- & z_+ \\ z_+ & z_+ \end{pmatrix}, \quad \det |B| = z_+ z_- = z^2. \quad (135)$$

One can easily check that the substitution of the Eqs. (135) into Eq. (134) gives $P(N_+, Q)$ from the Eq. (130).

C. Microcanonical Ensemble

The microcanonical partition function can be calculated in the same way. In the simplest case of massless Boltzmann particles we know the exact answer [27]:

$$P(N|E) = \frac{P(N, E)}{P(E)} = \frac{1}{Z^{MCE}} \frac{x^N}{E (3N-1)! N!}, \quad (136)$$

where $Z^{MCE} = \frac{x}{2E} {}_0F_3\left(\frac{4}{3}, \frac{5}{3}, 2; \frac{x}{27}\right)$ is the MCE partition function, ${}_0F_3$ is a generalized hypergeometric function, and $x \equiv gVE^3/\pi^2$. Let us consider $P(E)$ in Eq. (136). One can obtain it by straightforward iterative calculations similar to the Ref. [27]:

$$\begin{aligned} P(E) &= Z^{-1} \sum_{N=1}^{\infty} \frac{1}{N!} \frac{gV}{(2\pi)^3} \int d^3 p_1 \dots \frac{gV}{(2\pi)^3} \int d^3 p_N \exp\left[-\sum_{k=1}^N |\vec{p}_k|/T\right] \delta\left(E - \sum_{k=1}^N |\vec{p}_k|\right) \\ &= Z^{-1} \sum_{N=1}^{\infty} \frac{1}{N!} \left(\frac{gV}{2\pi^2}\right)^N \int_0^\infty p_1^2 dp_1 \dots \int_0^\infty p_{N-1}^2 dp_{N-1} e^{-E/T} \left(E - \sum_{k=1}^{N-1} p_k\right)^2. \\ &= \frac{e^{-E/T} Z^{MCE}}{Z} \end{aligned} \quad (137)$$

The saddle point expansion for $P(E)$ gives:

$$\begin{aligned} P(E) &= Z^{-1} \int_{-\infty}^{\infty} \frac{d\phi_E}{2\pi} \sum_{N=1}^{\infty} \frac{1}{N!} \frac{gV}{(2\pi)^3} \int d^3 p_1 \dots \frac{gV}{(2\pi)^3} \int d^3 p_N \exp\left[-\sum_{k=1}^N |\vec{p}_k|/T\right] e^{-i\phi_E(E - \sum_{k=1}^N |\vec{p}_k|)} \\ &= Z^{-1} \int_{-\infty}^{\infty} \frac{d\phi_E}{2\pi} e^{-iE\phi_E + \frac{\langle N \rangle}{(1-iT\phi_E)^3}} = Z^{-1} \int_{-\infty}^{\infty} \frac{d\varphi_E}{2\pi T} e^{-iE\phi_E + \langle N \rangle (1 + 3iT\varphi_E - 6T^2\varphi_E^2 + \dots)} \\ &\simeq P_G(E) = \frac{1}{\sqrt{8\pi T \langle E \rangle}} e^{-\frac{(E - \langle E \rangle)^2}{8T \langle E \rangle}}, \end{aligned} \quad (138)$$

where $\langle N \rangle = gVT^3/\pi^2$, $\langle E \rangle = 3\langle N \rangle T$ and $Z = e^{\langle N \rangle}$. The exact distribution $P(E)$ and its Gauss approximation $P_G(E)$, their ratio $R = P_G(E)/P(E)$, and the integrant $Int = Re[e^{(3\langle N \rangle - E/T)i\varphi_E}]$ (the imaginary part vanish because of symmetric integral bounds) are shown in the Figs. 6, 7 for

the parameters $E = 20 \text{ GeV}$, $V \simeq 768 \text{ fm}^3$ and for two different temperatures: the 'equilibrium' temperature $T = 160 \text{ MeV}$ that gives $\langle E \rangle = E$ and for 'non-equilibrium' temperature $T = 120 \text{ MeV}$ that gives $\langle E \rangle = 6.32 \text{ GeV} \neq E$.

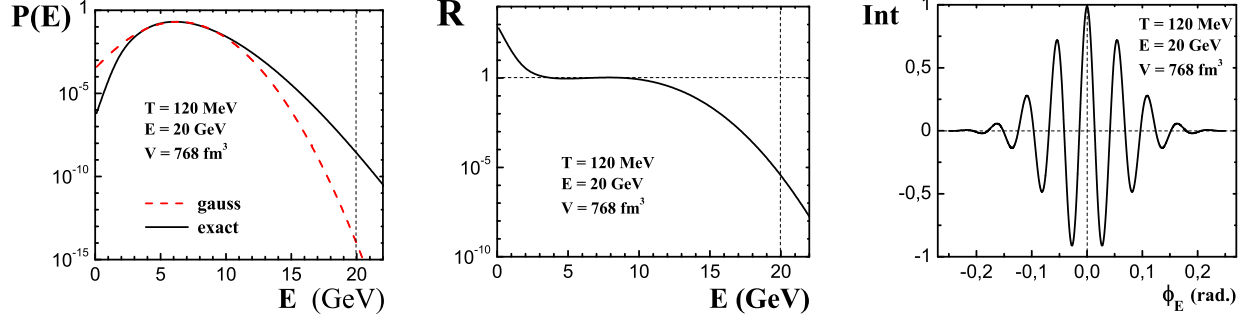


FIG. 6: The comparison of the exact distribution $P(E)$ and its Gauss approximation $P_G(E)$ (left), their ratio $R = P_G(E)/P(E)$ (center) and the integrant $Int = Re[e^{(3\langle N \rangle - E/T)i\phi_E}]$ for the parameters $\langle E \rangle = 20 \text{ GeV}$, $T = 120 \text{ MeV}$

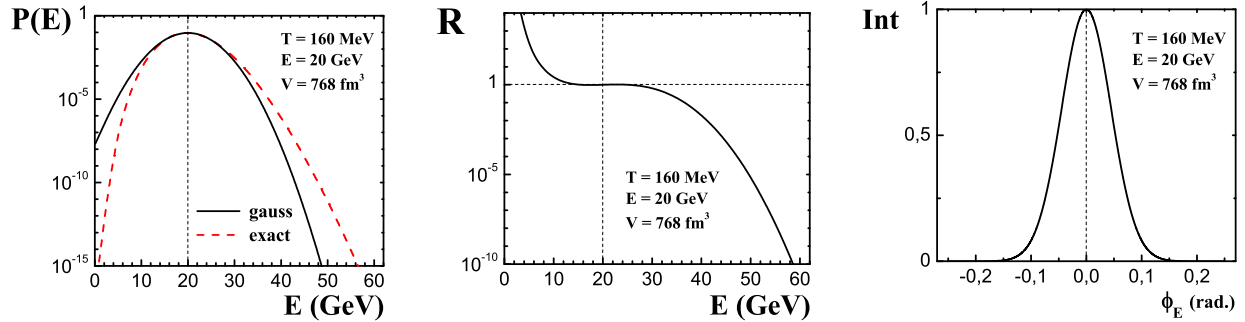


FIG. 7: The same as Fig. 6 but for $T = 160 \text{ MeV}$.

Comparing the Figs. 4, 5 and Figs. 6, 7 one can see that if one choose the wrong value of temperature than the integrant Int in $P(E)$ oscillate fast and Gauss approximation gives wrong result for the MCE system see Fig. 6. It means that if we calculate the MCE distribution $P(N|E)$ by means of the auxiliary GCE distributions $P(N, E)$ and $P(E)$ than the temperature play the role of the 'chemical potential' similarly to the CE.

D. Resonance Decay

In this subsection we would like to provide a connection between probability distribution P , cumulant generation function Ψ , and the generating function G [26] that was firstly used to

take into account resonance decays analytically. Let us consider again a simple example of an ideal Boltzmann gas of positively and negatively charged particles, i.e. particle-antiparticle gas, and, additionally, 'allow' the decays of positively charged particles as well as their antiparticles. The formula for $P(Q)$, (127), remains the same, while $P(N_+, Q)$, (129), changes:

$$P(N_+, Q) = \frac{1}{Z} \int_{-\pi}^{\pi} \frac{d\phi_Q}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi_+}{2\pi} \exp \left[-iQ\phi_Q - iN_+\phi_+ \right. \\ \left. + z \exp \left(\frac{\mu_Q}{T} + i\phi_Q \right) \sum_c \Gamma_+^c \exp(i c \phi_+) + z \exp \left(-\frac{\mu_Q}{T} - i\phi_Q \right) \sum_c \Gamma_-^c \exp(i c \phi_+) \right], \quad (139)$$

where c is a multiplicity of a particular decay channel and Γ_{\pm}^c is a branching ratio that corresponds to the decay of positively/negatively charged resonance.

There is no any real system that corresponds to the partition function (139), but it is the simplest example where we can show how these formulas work.

We can expand the exponent in the Eq. (139):

$$P(N_+, Q) = \frac{1}{Z} \int_{-\pi}^{\pi} \frac{d\phi_Q}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi_+}{2\pi} \exp \left[-i(Q - [n_+ - n_-]) \phi_Q - iN_+\phi_+ \right] \\ \times \sum_{n_+, n_-=0}^{\infty} \frac{z_+^{n_+} z_-^{n_-}}{n_+! n_-!} \left(\sum_c \Gamma_+^c \exp(i c \phi_+) \right)^{n_+} \left(\sum_c \Gamma_-^c \exp(i c \phi_+) \right)^{n_-} \quad (140)$$

and immediately recognize the generating function that was used in our previous work [26]:

$$G(\phi_+) \equiv \left(\sum_c \Gamma_+^c \exp(i c \phi_+) \right)^{n_+} \left(\sum_c \Gamma_-^c \exp(i c \phi_+) \right)^{n_-}. \quad (141)$$

The difference from the generation function used in [26] is that here we study the case with only two types of resonances, n_{\pm} , and we also use the different definition for a decay channel.

From the Eq. (140) one can also see that the cumulant generating function Ψ is connected with the generating function G in a very simple way:

$$\Psi(\phi_+) = \ln \left[\sum_{n_+, n_-=0}^{\infty} \frac{z_+^{n_+} z_-^{n_-}}{n_+! n_-!} G(\phi_+) \right]. \quad (142)$$

In order to make a comparison with the Ref. [26] one should also remember some technical moment that in the paper [26] we used the auxiliary parameters λ that are the same as $\exp(i\phi_+)$ in this paper, however

$$-i \frac{\partial}{\partial \phi_+} = \lambda_+ \frac{\partial}{\partial \lambda_+}, \quad \text{and} \quad \left(-i \frac{\partial}{\partial \phi_+} \right)^2 = \lambda_+ \frac{\partial}{\partial \lambda_+} \left(\lambda_+ \frac{\partial}{\partial \lambda_+} \right). \quad (143)$$

From the Eq. (142) it also can be easily seen that we can multiply the generation function by any other normalized distribution for example by the finite acceptance probability distribution and it will enter the partition function in the same way as the generating function G (see Section V B).

IX. SUMMARY AND CONCLUSION

We have presented an analytical expansion method for calculation of distributions at finite volume for the canonical as well as the microcanonical ensembles of the ideal relativistic hadron resonance gas. The introduction of temperature into the microcanonical partition function and chemical potentials into the canonical partition function have lead to the identification of the grand canonical partition function with the characteristic function of associated joint probability distributions. The microcanonical and canonical multiplicity distributions could thus be defined through the conditional probability distributions of finding a certain multiplicity while other parameters (global charge or energy) were taken to be fixed.

We have shown that in the thermodynamic limit the central region of the canonical and microcanonical partition functions can be approximated by multivariate normal distributions. Multiplicity distributions tend thus to Gaussians as system size is increased. In particular, we could find a simple formula for the scaled variance, hence the width of this Gaussian, in this limit. It was further possible to show the equivalence of our results to those obtained from the microscopic correlator approach.

Apart from providing a simplified formula for asymptotic multiplicity fluctuations, the approach presented in this work owns a few conceptual advantages. In considering finite volume corrections to the system partition function, thus relaxing the assumption of thermodynamic equivalence of different statistical ensembles, one is lead to demanding that the partition function should be maximized for a particular set of conserved charges. It turned out that this requirement is entirely equivalent to the well known textbook definitions of chemical potential in the canonical ensemble as the derivative of Helmholtz free energy with respect to conserved charge and temperature in the microcanonical ensemble through differentiation of entropy with respect to conserved energy.

Our method is based on Fourier analysis of the grand canonical partition function. Conventionally one would not introduce chemical potentials and temperature into these calculations.

However one then faces the problem of a heavily oscillating (or even irregular) integrant, making numerical integration unpractical. Artificially introduced temperature and chemical potentials, correctly chosen, produce a very smooth integrant allowing for expansion of the integrant in powers of volume. Analytical solutions to asymptotic multiplicity distributions could thus be found in terms of Laplace's expansion, while finite volume corrections could be obtained from Gram-Charlier expansion. A first comparison with available analytical solutions to simple statistical systems suggests that good results can be expected even for rather small volume. One drawback is that the results can only be applied to the central region of the distribution, owing to the fact that finite volume correction terms appear in the form of Hermite polynomial of low order.

Additionally we have included resonance decay directly into the system partition function of the hadron resonance gas model. This treatment has proven to be more economical than the previous handling through generating functions or Monte Carlo techniques. Neglecting correlation in momentum space a first order approximation to the effect of finite acceptance can be made.

This paper provides a connection between the two previously published methods for calculation of multiplicity distributions, the microscopic correlator approach and saddle point expansion, and should be of great practical use.

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APPENDIX A: CHARACTERISTIC FUNCTION

Let us consider a probability distribution of some observable χ which can take values $f_\chi(x)$ on the real x axis with the following normalization:

$$\int_{-\infty}^{\infty} dx f_\chi(x) = 1. \quad (\text{A1})$$

Hence $f_\chi(x)$ defines the probability of finding a value x for the observable χ . One can define the characteristic function $\Phi_\chi(i\theta)$ as the Fourier back transformation of the probability distribution $f_\chi(x)$:

$$\Phi_\chi(i\theta) \equiv \hat{f}_\chi(x) = \int_{-\infty}^{\infty} dx e^{ix\theta} f_\chi(x). \quad (\text{A2})$$

The moments $\mathcal{E}^n(\chi)$ can directly be calculated from the characteristic function from the respective derivatives at the origin:

$$i^n \mathcal{E}^n(\chi) \equiv \left. \frac{d^n \Phi_\chi(i\theta)}{d\theta^n} \right|_{\theta=0} = i^n \int_{-\infty}^{\infty} dx x^n f_\chi(x). \quad (\text{A3})$$

Once the characteristic function $\Phi_\chi(i\theta)$ is known, the probability distribution $f_\chi(x)$ can be found by Fourier transformation:

$$f_\chi(x) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{-ix\theta} \Phi_\chi(i\theta). \quad (\text{A4})$$

APPENDIX B: CE PARTITION FUNCTION

The (generalized) system partition function of a system with conserved charge vector $Q^j = (B, S, Q)$, Eq.(7), reads:

$$\mathcal{Z}^{Q^j} = \left[\prod_j \int_{-\pi}^{\pi} \frac{d\phi_j}{2\pi} \right] e^{-iQ^j \phi_j} \exp \left[\sum_l z_l(\phi_j, \mu_j) \right]. \quad (\text{B1})$$

The single particle partition function Eq.(8) includes chemical potentials as well as Wick rotated fugacities. Expanding the logarithm in Eq.(8) yields:

$$z_l(\phi_j) = \frac{g_l V}{(2\pi)^3} \int d^3 p \ln \left(1 \pm e^{-\frac{\varepsilon_l}{T}} e^{\frac{\mu_l}{T}} e^{iq_l^j \phi_j} \right)^{\pm 1} \quad (\text{B2})$$

$$= \sum_{n_l=1}^{\infty} (\mp 1)^{n_l+1} \frac{g_l V}{(2\pi)^3} \int d^3 p \frac{1}{n_l} e^{-\frac{n_l \varepsilon_l}{T}} e^{n_l q_l^j \left(\frac{\mu_l}{T} + i\phi_j \right)}, \quad (\text{B3})$$

where the effective chemical potential of particle species l is given by $\mu_l = q_l^j \mu_j$, and the single particle partition function of a ‘lump’ of mass n_l m_l of species l is given by [32] $z_{n_l} = \frac{g_l V}{(2\pi)^3} \int d^3p \, n_l^{-1} e^{-\frac{n_l \varepsilon_l}{T}}$. Eq.(B1) can be thus written as:

$$\mathcal{Z}^{Q^j} = \left[\prod_j \int_{-\pi}^{\pi} \frac{d\phi_j}{2\pi} \right] e^{-iQ^j \phi_j} \exp \left[\sum_l \sum_{n_l=1}^{\infty} (\mp 1)^{n_l+1} z_{n_l} e^{n_l q_l^j (\frac{\mu_j}{T} + i\phi_j)} \right] \quad (\text{B4})$$

$$= \left[\prod_j \int_{-\pi}^{\pi} \frac{d\phi_j}{2\pi} \right] e^{-iQ^j \phi_j} \prod_l \prod_{n_l=1}^{\infty} \sum_{k_{n_l}=0}^{\infty} \frac{((\mp 1)^{n_l+1} z_{n_l})^{k_{n_l}}}{k_{n_l}!} e^{k_{n_l} n_l q_l^j (\frac{\mu_j}{T} + i\phi_j)} \quad (\text{B5})$$

Only sets of numbers $\{k_{n_l}\}$ that meet the requirement:

$$\sum_l \sum_{n_l=1}^{\infty} k_{n_l} n_l q_l^j - Q^j = 0 \quad (\text{B6})$$

have a non-vanishing contribution to the integral. For any such set one finds:

$$\sum_l \sum_{n_l=1}^{\infty} k_{n_l} n_l q_l^j \frac{\mu_j}{T} = Q^j \frac{\mu_j}{T} . \quad (\text{B7})$$

Therefore one can pull the factor Eq.(B7) in front of the integral (B1):

$$\mathcal{Z}^{Q^j} = e^{Q^j \frac{\mu_j}{T}} \left[\prod_j \int_{-\pi}^{\pi} \frac{d\phi_j}{2\pi} \right] e^{-iQ^j \phi_j} \prod_l \prod_{n_l=1}^{\infty} \sum_{k_{n_l}=0}^{\infty} \frac{(z_{n_l})^{k_{n_l}}}{k_{n_l}!} e^{i k_{n_l} n_l q_l^j \phi_j} . \quad (\text{B8})$$

However this is just the standard definition of the system partition function times the factor Eq.(B7), thus:

$$\mathcal{Z}^{Q^j} = e^{Q^j \frac{\mu_j}{T}} Z^{Q^j} , \quad \text{and} \quad Z^{GCE} = \left[\prod_{j=1}^J \sum_{Q^j=-\infty}^{\infty} \right] \mathcal{Z}^{Q^j} . \quad (\text{B9})$$

A similar calculation can be shown to hold for the MCE.

APPENDIX C: MCE PARTITION FUNCTION

Having introduced temperature in the MCE partition is probably a somewhat unusual approach. Here we consider this in detail in the example of a massless gas in Boltzmann approximation without momentum conservation. We will first state and solve our version of the MCE partition function and compare to the analytical solution known from textbooks (e.g. [33]).

We adopt the notation of [27], $\mathcal{Z}^{E,N} = \mathcal{W}_N(E, V)$ and $\mathcal{Z}^E = \mathcal{W}(E, V)$. The number of states consistent with the constraints of fixed energy E and particle number N in a GCE system is given by the double Fourier integral over its GGPF $\mathcal{Z}(\phi_N, \phi_E)$:

$$\mathcal{W}_N(E, V, T) \equiv \int_{-\pi}^{\pi} \frac{d\phi_N}{2\pi} \int_{-\infty}^{\infty} \frac{d\phi_E}{2\pi} e^{-iN\phi_N} e^{-iE\phi_E} \exp \left[\frac{gV}{2\pi^2} \int_0^{\infty} p^2 dp e^{-\beta p} e^{ip\phi_E} e^{iN\phi_N} \right]. \quad (\text{C1})$$

Expanding the exponential and solving the integral over ϕ_N yields:

$$\mathcal{W}_N(E, V, T) = \frac{1}{N!} \left(\frac{gV}{2\pi^2} \right)^N \int_{-\infty}^{\infty} \frac{d\phi_E}{2\pi} e^{-iE\phi_E} \left[\int_0^{\infty} p^2 dp e^{-\beta p} e^{ip\phi_E} \right]^N. \quad (\text{C2})$$

Further now solving the integral over momentum gives:

$$\mathcal{W}_N(E, V, T) = \frac{1}{N!} \left(\frac{gV}{2\pi^2} \right)^N \int_{-\infty}^{\infty} \frac{d\phi_E}{2\pi} e^{-iE\phi_E} \left[\frac{2}{(\beta - i\phi_E)^3} \right]^N. \quad (\text{C3})$$

Eq. (C3) has obviously a pole of order $3N$ at $\phi_E = -i\beta$. So we close the integration over the lower hemisphere and use the residue theorem.

$$\begin{aligned} \mathcal{W}_N(E, V, T) &= \frac{-i}{N!} \left(\frac{gV}{\pi^2} \right)^N \text{Res} \left[\frac{e^{-iE\phi_E}}{(\beta - i\phi_E)^{3N}} ; \phi_E = -i\beta \right] \\ &= \frac{-i}{N!} \left(\frac{gV}{\pi^2} \right)^N \frac{1}{(3N-1)!} \lim_{\phi_E \rightarrow -i\beta} \left[\frac{d^{3N-1}}{d\phi_E^{3N-1}} (\phi_E + i\beta)^{3N} \frac{e^{-iE\phi_E}}{(\beta - i\phi_E)^{3N}} \right] \\ &= \left(\frac{gV}{\pi^2} \right)^N \frac{E^{3N-1}}{N! (3N-1)!} e^{-\beta E}. \end{aligned} \quad (\text{C4})$$

Multiplication with the inverse Boltzmann factor yields the result known from the literature [1, 27, 33], $W_N(E, V) \equiv \mathcal{W}_N(E, V, T) e^{\beta E}$:

$$W_N(E, V) = \left(\frac{gV}{\pi^2} \right)^N \frac{E^{3N-1}}{N! (3N-1)!}. \quad (\text{C5})$$

Using condition Eq.(98) it further follows that the temperature is given by the average energy per particle $T \equiv E/(3N-1)$. In complete analogy to what was presented before:

$$W(E, V) = \sum_{N=1}^{\infty} W_N(E, V), \quad \text{and} \quad \mathcal{W}(E, V, T) = \sum_{N=1}^{\infty} \mathcal{W}_N(E, V, T). \quad (\text{C6})$$

However in both case we define the MCE multiplicity distribution by:

$$P(N|E) = W_N(E, V) W^{-1}(E, V) = \mathcal{W}_N(E, V, T) \mathcal{W}^{-1}(E, V, T). \quad (\text{C7})$$

The introduction of temperature in Eq.(C1) does not necessarily simplify this calculation (it even drops out entirely from the r.h.s of Eq.(C7)), however makes approximations possible. The GCE partition function finally is given by:

$$Z^{GCE}(V, T) = 1 + \int_0^\infty dE W(E, V) e^{-\beta E} = 1 + \int_0^\infty dE \mathcal{W}(E, V, T) = \exp \left[\frac{gVT^3}{\pi^2} \right]. \quad (\text{C8})$$

The additional term ‘+1’ arises from the fact that the MCE partition function only includes states with at least one particle, while the state of zero energy and zero particle number contributes to the GCE partition function.

APPENDIX D: WIDTH AND NORMALIZATION

The inverse of a nonsingular $J \times J$ matrix $\tilde{\kappa}_2$ can be obtained from its adjoint and its corresponding minors [38]:

$$(\tilde{\kappa}_2^{-1})_{i,j} = \frac{\text{adj } \tilde{\kappa}_2}{\det |\tilde{\kappa}_2|} = \frac{[(\tilde{M})_{i,j}]^T}{\det |\tilde{\kappa}_2|}. \quad (\text{D1})$$

In our case only the minor $(\tilde{M})_{1,1} \equiv \det |\kappa_2|$ is of interest, hence we find for the element in the upper left corner of the inverse of $\tilde{\kappa}_2$:

$$(\tilde{\kappa}_2^{-1})_{1,1} = \frac{\det |\kappa_2|}{\det |\tilde{\kappa}_2|}. \quad (\text{D2})$$

Generally we can find the inverse of a matrix by multiplication of a diagonal matrix, with the inverses eigenvalues on its diagonal $\tau_{a,b}^{-1}$, with orthonormal transformation matrices, which can be formed from the eigenvectors \vec{v}_a of $\tilde{\kappa}_2$,

$$(\tilde{\kappa}_2^{-1})_{i,j} = \sum_{a=1}^J \sum_{b=1}^J v_{i,a} (\tau^{-1})_{a,b} v_{b,j}^T, \quad (\text{D3})$$

where $v_{i,a}$ is the a^{th} component of the i^{th} eigenvector, $v_{b,j}^T$ is the transpose of $v_{i,a}$, hence a matrix with eigenvectors in its columns, and $(\tau^{-1})_{a,b}$ is a matrix with the corresponding inverse of the a^{th} eigenvalues t_a^{-1} on its diagonal. We find for the upper left most entry:

$$(\tilde{\kappa}_2^{-1})_{1,1} = \sum_{a=1}^J v_{1,a}^2 t_a^{-1} = \frac{\det |\kappa_2|}{\det |\tilde{\kappa}_2|}. \quad (\text{D4})$$

Likewise, we can express the inverse sigma tensor in term of eigenvalues and eigenvectors of $\tilde{\kappa}_2$ (see Section VII):

$$(\tilde{\sigma}^{-1})_{i,j} = \sum_{a=1}^J \sum_{b=1}^J v_{i,a} (\tau^{-1/2})_{a,b} v_{b,j}^T = \sum_{a=1}^J v_{i,a} v_{j,a} t_a^{-1/2}. \quad (\text{D5})$$

We just need the sum of squares of the entries of the left most column:

$$\sum_{i=1}^J (\tilde{\sigma}^{-1})_{i,1}^2 = \sum_{i=1}^J \sum_{a=1}^J \sum_{b=1}^J v_{i,a} v_{1,a} t_a^{-1/2} v_{i,b} v_{1,b} t_b^{-1/2} = \sum_{a=1}^J v_{1,a}^2 t_a^{-1}, \quad (\text{D6})$$

since $\sum_{i=1}^J v_{i,a} v_{i,b} = \delta_{a,b}$. This coincides with Eq. (D4) and proofs Eq.(27),

$$\sum_{i=1}^J (\tilde{\sigma}^{-1})_{i,1}^2 = \frac{\det |\kappa_2|}{\det |\tilde{\kappa}_2|}, \quad (\text{D7})$$

in the most general case of a J dimensional 2^{nd} rank tensor $\tilde{\kappa}_2$.

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